CONCISE NOTES

ON

DEVELOPING ALGEBRAIC THINKING

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$$

EMA 111S

Unit 1 Components of Algebraic Thinking

Introduction

For many people, the thought of studying algebra conjures up memories of 'an intensive study of the last three letters of the alphabet'. While this description of algebra may be representative of most people's high school experiences with the subject, much of algebra does not involve problems with *x*, *y*, or *z*. **Algebra is, in essence, the study of patterns and relationships;** ***finding the value of x or y in an equation is only one-way to apply algebraic thinking to a specific mathematical problem.***

In fact, the potential for students to think algebraically resides in many of the arithmetic problems they regularly do in upper elementary school; it requires only a shift in language or a slight extension of a basic arithmetic problem to open up the space of algebraic thinking for students. Algebraic problems in elementary school do not have to include the dreaded phrase, "Solve for *x*." Considering the role of algebra in grades 3 – 5 requires us to go beyond the limited definition of "problems with letters" to a generative view of algebraic thinking.

What is algebraic thinking?

John Van de Walle, (2004), who writes, gives a useful definition of algebraic thinking: "*Algebraic thinking* involves representing, generalizing, and formalizing patterns and regularity in all aspects of mathematics." (p. 417). As we think about algebraic thinking, it may also help to define the term *algebra*. The NCTM *Principles and Standards for School Mathematics* (2000) includes a description of algebra that goes beyond manipulating symbols. In the *Standards*, algebra is defined as:

- Understanding patterns, relations, and functions;
- Representing and analyzing mathematical situations and structures using algebraic symbols;
- Using mathematical models to represent and understand quantitative relationships and
- Analyzing change in various contexts

Although it may include variables and expressions, algebraic thinking has a broader and different connotation than the term algebra. The term **algebraic thinking can be defined as "the use of any of a variety of representations that handle quantitative situations in a relational way.** Another definition of **algebraic thinking is "the ability to operate on an unknown quantity as if the quantity was known, in contrast to arithmetic thinking which involves operations on known quantities.** The **algebraic thinking could be considered to be the "capacity to represent quantitative situations so that relations among variables become apparent".** These definitions are all similar, and we use them to guide our own use of the term "algebraic thinking" (Steele & Johanning, 2004: 65). Algebraic thinking consists of more than just learning how to solve for the variables x and y; it helps students think about mathematics at an abstract level, and provides them with a way to reason about real-life problems. **You can explore three components of algebraic thinking: (1) making generalizations, (2) conceptions about the equals sign (equality), and (3) thinking about unknown quantities.**

In this course, we will consider three distinct aspects of algebraic thinking that can be identified in elementary mathematics instruction: *generalization, concepts of equality, and thinking with unknown quantities.* These three components of algebraic thinking provide a useful framework for recognizing whether students in grades 3 through 5 are thinking algebraically, and for determining whether a problem can be viewed algebraically.

Generalisation

Prominent in most definitions of algebra is the notion of "patterns." The ability to discover and replicate mathematical patterns is important throughout mathematics. The authors of the *Principles and Standards for School Mathematics* talk extensively about the important role that understanding patterns plays in algebraic thinking: In grades 3–5, students should investigate numerical and geometric patterns and express them mathematically in words or symbols. They should analyze the structure of the pattern and how it grows or changes, organize this information systematically, and use their analysis to develop generalizations about the mathematical relationships in the pattern. Young students can have meaningful experiences with generalizing about patterns, even though they do not usually express their mathematical ideas using variables and standard functions. For example, when exploring a pattern such as 1, 3, 5, 7, 9,…, young students may make the following observations:

1. "If you add 1 to an even number, you always get an odd number"
2. "If you add 2 to an odd number, you always get another odd number"
3. "If you start at 1 and keep adding 2, you get all the odd numbers"
4. "If you can separate a number into two equal groups, it's an even number. If one's left over, it's an odd number."

All of these observations are ways of thinking about a simple pattern—the progression of positive odd integers. However, they also provide evidence of algebraic thinking, because each description relies on some sort of generalization that can be applied to any number. For example, notice how observation 1 contains the term "an even number." The student here is generalizing that no matter how large or small the even number, adding 1 will create an odd number. Likewise, in observation 4, the student has identified the property that any even number can be split into even groups, but odd numbers cannot. Both of these observations are examples of generalization, since they are **projecting a mathematical property onto a whole category of numbers**; in this case, "the even numbers." It may take some time for students to develop strategies for justifying a pattern. The **first steps are noticing that there is a pattern** in a number sequence, and then wondering if that pattern continues as the numbers get larger. **Describing the pattern is the next step, followed by extending it**. Eventually, students will arrive at a generalized understanding of the pattern; they will be able to predict whether a specific number (or term) is part of a pattern without calculating each consecutive term. For example, given the pattern 1, 3, 5, 7, 9, …, above, students will be able to determine that a number such as 381 is part of the pattern because it is an odd number, and will not need to write out each odd number from 1 to 381 to be convinced of this fact. In upper elementary school, most students will be ready to work on proving statements such as "adding 2 to an odd number produces another odd number," but their ideas about proof will continue to evolve as they expand upon them in senior high svhools. From a formal algebraic perspective, all four statements above follow from the fact that all odd numbers are of the form $2n+1$, but students can make and test conjectures long before they ever see such an expression. It is important to keep in mind that as students propose generalizations such as those above, they may be basing their claims on only one or two instances of a pattern. Mathematically this is not enough evidence to determine whether a pattern exists. In observation 3 above, for example, a student may have noticed that adding $1 + 2 = 3$ (adding 2 to an odd number produced another odd number), and adding $3 + 2 = 5$, also an odd number. However, she may not have investigated any numbers beyond those. **It is important for elementary students to learn that forming generalizations from only a few instances can lead to inaccurate conclusions.**

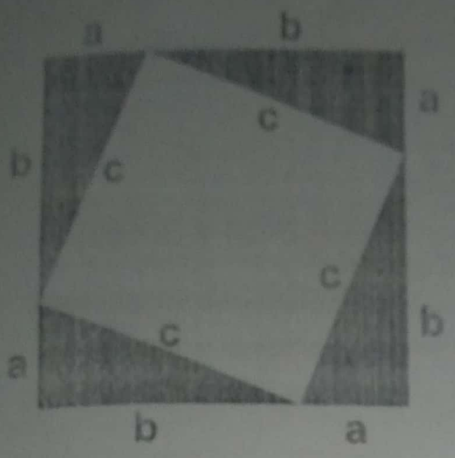
One example of this can be seen in students' solutions to the "Frog in the Well". A frog climbs 3feet in the first hour and 2 feet for each subsequent 2hours. How long will it take the frog to reach a height of 10-foot well?

The students try to use generalization to solve this problem, and figure that the frog climbs 2 feet total for each 2-hour period because he climbs up three feet in the first hour and slips down one in the second hour. Using this generalization, they come to the conclusion that it will take a frog 10 hours to reach the top of a 10-foot well. However, while the relationship holds in general for each 2-hour period, the ninth hour occurs in the middle of a 2- hour period. During this hour the frog reaches the top of the well and climbs out, and consequently does not "slide down." While students have made a generalization that is true in most cases, they have neglected to notice that their current problem is an exception to the general rule of up three, down two. In the end, their understanding of the relationship actually misleads them into solving the problem incorrectly.

Proof of the Pythagorean Theorem using Algebra

We can show that $\mathbf{a^2 + b^2 = c^2}$ using Algebra

Take a look at this diagram, it has that "abc" triangle in it (four of them actually):



Area of Whole Square

It is a big square, with each side having a length of a+b, so the total area is:
$A = (a+b)(a+b)$

Area of the Pieces

Now let's add up the areas of all the smaller pieces:
First, the smaller (tilted) square has an area of: c^2
Each of the four triangles has an area of: $\frac{1}{2}ab$
So all four of them together is: $4 \cdot \frac{1}{2}ab = 2ab$
Adding up the tilted square and the 4 triangles gives: $A = c^2 + 2ab$
Both Areas Must Be Equal

The area of the **large square** is equal to the area of the **tilted square** and the **4 triangles**. This can be written as:

$(a+b)(a+b) = c^2 + 2ab$

NOW

let us rearrange this to see if we can get the pythagoras theorem:
Start with: $(a+b)(a+b) = c^2 + 2ab$
Expand $(a+b)(a+b)$
$a^2 + 2ab + b^2 = c^2 + 2ab$
Subtract "2ab" from both sides: $a^2 + b^2 = c^2$

DONE

Proving that 1+2+3+...+n is n(n+1)/2

We give three proofs here that the n-th Triangular number, 1+2+3+...+n is n(n+1)/2. The first is a visual one involving only the formula for the area of a rectangle. This is followed by two proofs using algebra. The first uses "..." notation and the second introduces you to the Sigma notation which makes the proof more precise.

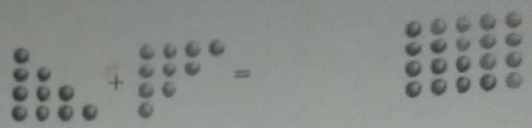
A visual proof that 1+2+3+...+n = n(n+1)/2
We can visualize the sum 1+2+3+...+n as a **triangle of dots**. Numbers which have such a pattern of dots are called **Triangle (or triangular) numbers**, written T(n), the sum of the integers from 1 to n :

N	1	2	3	4	5	6
T(n) as a sum	1	1+2	1+2+3	1+2+3+4	1..5	1..6
T(n) as a triangle					...	
T(n)=	1	3	6	10	15	21

For the proof, we will *count the number of dots in T(n)* but, instead of *summing the numbers 1, 2, 3 etc up to n* we will find the total using only *one multiplication and one division!*

To do this, we will fit **two copies of a triangle of dots together**, one red and an upside-down copy in green.

E.g. T(4)=1+2+3+4



Notice that

- we get a **rectangle** which has the same number of rows (4) but has one extra column (5)
- so the rectangle is **4 by 5**
- it therefore contains 4x5=20 balls
- but we took **two** copies of T(4) to get this
- so we must have 20/2 = 10 balls in T(4), which we can easily check.

This visual proof applies to any size of triangle number.

Here it is again on T(5):

So T(5) is half of a rectangle of dots 5 tall and 6 wide, i.e. half of 30 dots, so T(5)=15.

Here's how a mathematician might write out the above proof using algebra:

$T(n)+T(n) = 1 + \quad 2 + \quad 3 + \quad \ldots + (n-1) + \quad N$

$\quad + n + \quad (n-1) + \quad (n-2) + \quad \ldots + 2 + \quad 1$ *Two copies, one red and the other, reversed, in green*

$= (1+n) + \quad (2+n-1) + \quad (3+n-2) + \quad \ldots + (n-1+2) + \quad (n+1)$ *pair off the terms, a red with a green*

$= (n+1) + \quad (n+1) + \quad (n+1) + \quad \ldots + (n+1) + \quad (n+1)$ *All the n pair-sums are equal to (n+1)*

$2\,T(n) = n\,(n+1)$

$T(n) = n\,(n+1)/2$

Using the Sigma notation

Some people regard the ‘’ as too vague and want a more precise alternative. For this reason, in summing a series, the **sigma notation** is used. **Sigma** is the name of the greek letter for the English "s", written as (like an M on its side) as a capital letter and (like a small b that's fallen over) in lower case. In this case, the "s" stands for "sum". (A tall curly form of S gives the mathematical symbol for **integration** - another kind of sum).

Mathematicians use the capital sigma for the sum of a series as follows:

$$\sum_{i=\text{starting value}}^{i=\text{final value}} i$$

- **a formula describes the i^{th} term** of the series being summed. It is written *after* the sigma;
- the starting value for i is written *below* the sigma;
- the ending value for i is written *above* the sigma

In fact, the formula after the sigma can be written in terms of *any variable not just i*, for instance k, but then we must indicate which is the letter that varies in the sum under the sigma. Often the variable is omitted *above* the sigma but *never omitted* below the sigma. Here are some examples:

The sum $10^2+11^2+12^2$, where the numbers added are the **square numbers** i^2:

$$\sum_{i=10}^{i=12} i^2$$

The same sum can also be written in many other ways, for instance, as the sum of the **square numbers** $(i+9)^2$ where this time i goes from 1 to 3

$$\sum_{i=1}^{i=3} (i+9)^2$$

or as the sum of the **square numbers** $(i+11)^2$ where this time i goes from -1 to 1 (i.e. i = -1, 0 and 1)

$$\sum_{i=-1}^{i=1} (i+11)^2$$

The sum 1+2+3+..+9 is T(9) or

$$T(9) = \sum_{i=1}^{i=9} i$$

Here is T(n) which is 1+2+3+...+n, this time omitting the second use of the i above the sigma:

$$\sum_{i=1}^{n} i = T(n)$$

and this time, we have T(n) but written backwards: n+ (n-1) + ... 3 + 2 + 1 where the i^{th} term is now n+1–i for i from 1 to n:

$$\sum_{i=1}^{i=n} (n+1-i) = T(n)$$

Finally, note that if all the terms are *independent of the variable*, for instance if there is no *i* in the formula but the variable below the sigma is *i*, then all the terms are *constant*. The *number of terms* will be given by the starting and ending values. Here, all the terms are fixed (constant) at 3:

$$\sum_{i=4}^{i=7} 3 = 3+3+3+3 = 12$$

Here is the algebraic proof from above but now written using the sigma notation:

T(n)+T(n) = i=n i + i=n (n+1–i) Two copies, one red and the other, reversed, in green i=1 i=1

$$2\,T(n) = \sum_{i=1}^{i=n} (i + n + 1 - i)$$ *pair off the terms, a red with a green*

$$2\,T(n) = \sum_{i=1}^{i=n} (n+1)$$ *n copies of (n+1): the i does not appear in the formula so all the terms are the same*

$$2\,T(n) = n\,(n+1)$$

$$T(n) = n\,(n+1)/2$$

Equality – the meaning of the "=" sign

Elementary texts sometimes hint at the relationships between arithmetic and algebra by noting that the problem "add 5 + 24" could just as well be stated "5 + 24 = ?" or "5 + 24 = ❑" or even "5 + 24 = *x*." While these notations create a connection between arithmetic and the "missing value" image of algebra, students can also be misled by the implications of these expressions. For example, consider the algebraic statement "5 + 24 = ? + 15." On the face of it, this expression is similar to the previous ones (e.g. 5 + 24 = ?) but there is one very important difference: the number that replaces the ? is no longer 29, but a smaller number that when added to 15, produces 29. However, when faced with a problem like this, research has shown that many elementary students persist in saying the answer is either the sum of the two addends to the left of the equals sign, or the sum of all the addends in the problem, regardless of their placement relative to the equals sign (Falkner, Levi, & Carpenter, 1999). Consequently, given the problem "5 + 24 = ? + 15," most elementary students would respond that the missing number was either 29 or 44. What's going on here? The issue resides in the meaning students assign to the "=" sign. In the case of the problem "5 + 24 = ?" the "=" can be thought of as **"the result of the previous computation."** That is a sufficient interpretation in this problem. However, in the example "5 + 24 = ? + 15," the equals sign must be interpreted differently. It is now a **statement of equivalence between two quantities**, in this case between "5 + 24" and "? + 15." Now it is clearer that the ? must be replaced by something other than 29, since "5 + 24" and "29 + 15" are not equivalent. ***Understanding that the sign "=" requires that one side of the expression be equivalent to the other is a basic tenet of algebra.*** Students will be stretching their algebraic thinking skills if they see a variety of problems with unknowns in different positions, such as:

- 4 + ? = 17 + 2
- ? + 15 = 12 + 32
- 13 + 24 = 50 + ?

(Note that the 3rd problem has a negative integer as a value for ?, which may or may not be appropriate for your students.)

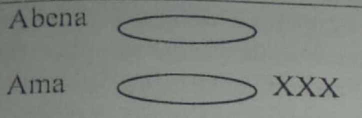
Unknown quantity

Besides the word **"variable," "unknown"** is one of the words most frequently associated with algebra. Along with this concept comes the idea that the "unknown" will eventually become "known;" this is what solving equations is usually about. But it's possible (and important) for students to work with expressions that include a variable that remains unknown. Most number tricks of the form, "choose a number, multiply it by 3, add 6, divide by 3, subtract 2 and tell me the number – and I'll tell you your original number," can be expressed algebraically without the need to use a specific number. The algebraic component is that the trick works for all numbers, not just a specific one for which we have to solve.

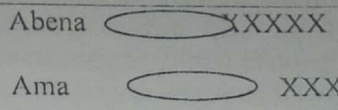
Here's an example of a problem with an unknown quantity that remains unknown. This problem is appropriate for students in grades 3 through 5: Suppose Abena has some number of pieces of erasers in her bowl. Ama has 3 more pieces of erasers than Abena has. Abena's mother gives her 5 more pieces of erasers. Now who has more? How many more? Then Abena gives Ama one of her pieces of erasers. Now who has more? How many more?

Students can solve this problem without creating algebraic expressions that contain variables. They may draw a picture to represent the number of erasers Abena has (e.g. a circle), and then represent Ama's erasers with a circle and three extra X's. They could then manipulate the pictures without ever specifying what is in the circle.

In this problem, finding the exact amount of erasers Abena has is not important, since the problem asks for a comparison between two quantities.



In this diagram, the amount of erasers that Abena and Ama begin with is represented as ovals. The extra pieces that Ama has are represented as x's. This diagram shows that Ama has more erasers, since he has 3 x's, and Abena has none.



In this diagram, Abena has been given 5 more pieces of erasers, which are represented by x's. Since she has more x's (or individual pieces of erasers) than Ama, she must also have more total erasers, because the quantities in the ovals are the same.

However, some problems similar to the one above cannot be solved without figuring out the value of an unknown number of erasers. For example: Suppose Abena has some number of pieces of erasers in her bowl. Ama has 3 more pieces of erasers than Abena has. If Abena gets more erasers so that she has twice as many as before, who has more erasers now? How many more? There isn't a single answer to this problem; it depends on how many erasers Abena had to begin with. So, if Abena had 2 erasers originally, Abena will now have 4, and Ama will have 5. On the other hand, if Abena begins with 5 erasers, then she will now have 10, while Ama has 8. The difference between these two kinds of problems is subtle, but as students approach fifth grade, they should be able to start making the distinction and solving them appropriately. These types of problems help develop algebraic thinking skills because they require students to think flexibly about quantities, and to learn how to compare related quantities. They also promote the idea that the relationship between two quantities (here, whether Abena or Ama has more erasers) can change depending on how the original amount is acted upon.

Getting ready for formal algebra

If students have experiences with all three kinds of algebraic thinking tasks in elementary school, they will come to the "formal" study of algebra in Junior and Senior high school armed with an already developed ability to reason algebraically. For example, as they encounter more complex linear equations in senior high svhools, students will be able to interpret the "=" *sign as an indication of equality, not as a sign requiring them to compute something.* They will have already considered the kinds of patterns that they may now be asked to express in algebraic form. And they will be prepared to work flexibly with variables as unknown quantities rather than needing to figure out its value immediately. With these insights in hand, students will find that algebra is not a mystery, but a territory that already has familiar landmarks.

Unit 2

Teaching Algebra of Sets, Relations, Mapping and Functions, Equivalent Relations and Properties of Integers

Introduction

Sets are the most basic of all mathematical objects, **simply collections of objects or a well-defined collection of objects** called elements or members of the set. Other synonyms are common for the word "set," such as collection, class, family, and ensemble. Hence, we can refer to a collection of people, family of fish, an ensemble of voters, and so on. We shall even consider sets whose members themselves are sets, such as the set of all classes at a university, or the family of all open intervals (a b,) on the real number line. If a set does not contain too many members, one can specify the set by simply writing down the members inside a pair of brackets, such as {John, Abu, Danaa} or by {2, 3, 5, 7, 13, 17, 19, 31, 61, 89} which contains the first ten Mersenne primes. Sometimes sets contain an infinite number of elements, like the natural numbers where we might specify them by {1,2,3, ...} , where the three dots after the 3 signify "and so on" and denotes the fact that the sequence of numbers is never ending.

One can also specify a set by specifying defining properties of the member of the set, such as

$\{x \in D : K(X)\}$

For all $x \in D$ such that K(X) holds which reads "the set of all x in a set A such that condition K (x) is true." The set of even integers could be denoted by {x∈ N : x is an even integer} . In the case when it is clear we are talking about natural numbers, we might simply write {x: x is an even integer}.

Specification of sets

There are three main ways to specify a set:

1. by listing all its members (list notation)
2. by stating a property of its elements (predicate notation);
3. by defining a set of rules which generates (defines) its members (recursive rules)

List notation

The first way is suitable only for finite sets. In this case we list names of elements of a set, separate them by commas and enclose them in braces: Examples: {1, 12, 45}, {George Washington, Bill Clinton}, {a,b,d,m}. "Three-dot abbreviation": {1,2, ..., 100}. {1,2,3,4,...} – this is not a real list notation, it is not a finite list, but it's common practice as long as the continuation is clear. Note that we do not care about the order of elements of the list, and elements can be listed several times. {1, 12, 45}, {12, 1, 45,1} and {45,12, 45,1} are different representations of the same set (see below the notion of identity of sets).

Predicate notation

Example: {x:x is a natural number and x < 8} Reading: "the set of all x such that x is a natural number and is less than 8" So the second part of this notation is a property the members of the set share (a condition or a predicate which holds for members of this set). Other examples: {x: x is a letter of Russian alphabet} {y: y is a student of UMass and y is older than 25} General form: { x: P(x)}, where P is some predicate (condition, property). The language to describe these predicates is not usually fixed in a strict way. But it is known that unrestricted language can result in paradoxes. Example: { x:x∉ x}. ("Russell's paradox") -- see the historical notes about it on pp 7-8. The moral: not everything that looks on the surface like a predicate can actually be considered to be a good defining condition for a set. Solutions – type theory, other solutions; we won't go into them.

Recursive rules

Example – the set E of even numbers greater than 3:

a) 4 ∈ E

b) if x ∈ E, then x + 2 ∈ E

c) nothing else belongs to E.

The first rule is the basis of recursion, the second one generates new elements from the elements defined before and the third rule restricts the defined set to the elements generated by rules a and b. (The third rule should always be there; sometimes in practice it is left implicit. It's best when you're a beginner to make it explicit.)

Identity and cardinality

Two sets are identical if and only if they have exactly the same members. So A = B iff (if and if) for every x, x ∈ A ⇔ x ∈ B. For example, {0, 2, 4} = {x| x is an even natural number less than 5}

From the definition of identity follows that there exists only one empty set; its identity is fully determined by its absence of members. Note that empty list notation {} is not usually used for the empty set, we have a special symbol Ø for it. **The number of elements in a set A is called the cardinality of A** written|A|. The cardinality of a finite set is a natural number. Infinite sets also have cardinalities but they are not natural numbers

Subsets

A set 'A' is a subset of a set B iff every element of 'A' is also an element of B. Such a relation between sets is denoted by A ⊆ B.

If A ⊆ B and A ≠ B we call A a proper subset of B and write A ⊂ B. (Caution: sometimes ⊂ is used the way we are using ⊆.) Both signs can be negated using the slash / through the sign. Examples: {a,b} ⊆ {d,a,b,e} and {a,b} ⊂ {d,a,b,e}, {a,b} ⊆ {a,b}, but {a,b} ⊄ {a,b}. Note that the empty set is a subset of every set. Ø ⊆ A for every set A. Why? Be careful about the difference between "member of" and "subset of".

Power sets

The set of all subsets of a set A is called the power set of A and denoted as ℘(A) or sometimes as 2A . For example, if A = {a,b}, ℘(A) = {Ø, {a}, {b}, {a,b}}. From the example above: a ∈ A; {a} ⊆ A; {a} ∈ ℘(A) Ø ⊆ A; Ø ∉ A; Ø ∈ ℘(A); Ø ⊆ ℘(A). Pay close attention to the definitions and it should come out all right. But if you don't pay close attention to the definitions, it's easy to make mistakes. Make sure you understand these examples before you try it. (But do try; and if you don't get something right the first time, we give you a chance to redo it.) 1.6. Operations on sets: union, intersection. We define several operations on sets. Let A and B be arbitrary sets. The union of A and B, written A ∪ B, is the set whose elements are just the elements of A or B or of both. In the predicate notation the definition is A ∪ B =def { x: x ∈ A or x ∈ B} Examples. Let K = {a,b}, L = {c,d} and M = {b,d}, then K ∪ L = {a,b,c,d} K ∪ M = {a,b,d} L ∪ M = {b,c,d}

(K ∪ L) ∪ M = K ∪ (L ∪ M) = {a,b,c,d} K ∪ K = K K ∪ Ø = Ø ∪ K = K = {a,b}. There is a nice method for visually representing sets and set-theoretic operations, called Venn diagrams. Each set is drawn as a circle and its members represented by points within it. The diagrams for two arbitrarily chosen sets are represented as partially intersecting – the most general case – as in Figure 1–1 p.13. The region designated '1' contains elements which are members of A but not of B; region 2, those members in B but not in A; and region 3, members of both B and A. Points in region 4 outside the diagram represent elements in neither set.

Functions

What is a function?

WHEN ONE THING DEPENDS on another, as for example the area of a circle depends on the radius -- in the sense that when the radius changes, the area will change -- then we say that the first is a "function" of the other. The area of a circle is a function of -- it depends on -- the radius.

Mathematically:

> *A rule that relates two variables, typically x and y, is called a function if to each value of x the rule assigns one and only one value of y. When that is the case, we say that y is a function of x.*

Thus a "function" must be single-valued ("one and only one"). For example, $y = 2x + 3$.

To each value of x there is a unique value of y.

Note: Every functtion is a relation, but not all relations are functions.

Domain and range

The values that x may assume are called the domain of the function. We say that those are the values for which the function is defined. In the function $y = 2x + 3$, the domain may include all real numbers. x could be any real number. Or, as in Example 1 below, the domain may be arbitrarily restricted.

There is one case however in which the domain must be restricted: A denominator may not be 0. In this function,

$$y = \frac{1}{x-2},$$

x may not take the value 2. For, division by 0 is an excluded operation.

Once the domain has been defined, then the values of y that correspond to the values of x are called the range. Thus if 5 is a value in the domain of $y = 2x + 3$, then

$13 = (2 \cdot 5) + 3)$ is the corresponding value in the range.

By the value of the function we mean the value of y. And so when $x = 5$, then we say that the value of the function $y = 2x + 3$, is 13. The range is composed of the values of the function. It is customary to call x the independent variable because we are given, or we must choose, the value of x first. y is then called the dependent variable because its value will depend on the value of x.

Example 1. Let the domain of a function be this set of values:

A = {0, 1, 2, −2} and let the variable x assume each one. Let the rule that relates the value of y to the value of x be the following:

$y = x^2 + 1$.

a) Write the set of ordered pairs (x, y) which "represents" this function.

Answer. {(0, 1), (1, 2), (2, 5), (−2, 5)}

That is, when $x = 0$, then $y = 0^2 + 1 = 1$.

When $x = 1$, then $y = 1^2 + 1 = 2$. And so on.

b) Write the set B which is the range of the function.

Answer. B = {1, 2, 5, 5}. The values in the range are simply those values of y that correspond to each value of x.

Notice that to each value of x in the domain there corresponds one -- and only one -- value of the function. Even though the value 5 is repeated, it is still one and only one value.

Example 2. Here is a relationship in which y is not a function of x:

$y^2 = x$

When $x = 4$, for example -- $y^2 = 4$ -- then $y = 2$ or -2. To each value of x, there is more than one value of y.

Problem 1. Let y be a function of x as follows:

$y = 3x^2$

a) Which is the independent variable and which the dependent variable?

x is the independent variable, y is the dependent.

b) The domain of a function are the values of the independent variable, which are the values of x.

c) What is the natural domain of that function?

Since there is no natural restriction on the values of x, the natural domain of that function is any real number. x could take any value on the x-axis.

d) The range of a function are the values of the dependent variable, which are the values of y.

e) What is the range of that function? (Consider that the values of x^2 are never negative.) $y \geq 0$

f) Write any three values of that function as members of an ordered pair.

For example, (1, 3), (2, 12), (3, 27)

Functional notation

The *argument* of the function Say that we are considering two functions -- two rules for determining y: $y = x^2 + 1$ and $y = 5x$.

Then it will be convenient to give each of them a name. Let us call the function -- the rule -- $y = x^2 + 1$ by the name "f" and let us call $y = 5x$ by the name "g." We will write the following:

$f(x) = x^2 + 1$ and $g(x) = 5x$.

We read this,

"f of x equals $x^2 + 1$ and g of x equals $5x$."

The parentheses in $f(x)$ ("f of x") do not mean multiplication. They are part of what is called *functional notation*. f is the *name* of the function and whatever appears within the parentheses is called the argument of the function. It is upon the argument that the function called f will "operate."

Thus, the function f has been defined as follows:

$f(x) = x^2 + 1$.

This means that the function f will *square* its argument, and then *add 1*.

For example,

$f(7) = 7^2 + 1 = 50$.

$f(-4) = (-4)^2 + 1 = 17$.

$f(t) = t^2 + 1$.

$f(x + h) = (x + h)^2 + 1 = x^2 + 2xh + h^2 + 1$.

The function f having been defined, that is how it will operate on any argument -- which is the input. The output is the value of the function. We could illustrate it as follows:

An argument x goes into the f machine. Out comes $x^2 + 1$

We write

$y = f(x)$.

"y is a function of x, whose name is f."

$f(x)$, then, is the dependent variable. Its value will depend on the value of x. We saw above that when $x = 7$, $f(x) = 50$. When $x = -4$, $f(x) = 17$. $f(x)$ is the dependent variable.

A function of a function

Again, let us consider these functions: $f(x) = x^2 + 1$ and $g(x) = 5x$.

And now consider this function, $f(g(x))$. "f of g of x" f has g as its argument:

$f(g(x)) = f(5x) = (5x)^2 + 1 = 25x^2 + 1$. Again, f squares its argument and adds 1. Now let's look at $g(f(x))$. g will operate on f. What does g do to its argument? It simply multiplies the argument by 5. Therefore, $g(f(x)) = g(x^2 + 1) = 5(x^2 + 1) = 5x^2 + 5$.

The parentheses in $g(x^2 + 1)$ are the parentheses of the functional notation. The parentheses of $5(x^2 + 1)$, however, are the grouping parentheses, which here indicate multiplication by 5.

Problem . *Read* each symbol.

a) $f(x)$ **"f of x"** b) $g(x)$ **"g of x"**

c) $f(2)$ **"f of 2"** d) $g(-1)$ **"g of -1"**

e) $f(x^2 - 1)$ **"f of $x^2 - 1$"** f) $f(g(x))$ **"f of g of x"**

Problem Let $f(x) = x^2 - 1$. Evaluate the following.

a) $f(1)$ **$1^2 - 1 = 0$** b) $f(-2)$ **3**

c) $f(2/3)$ **$-5/9$** d) $f(-7/5)$ **24/25**

Problem Let $g(x) = 2 - x$. Evaluate the following.

a) $g(0)$ **2** b) $g(-1)$ **3**

c) $g(6)$ **-4** d) $g(-4)$ **6**

Problem Let $y = f(x) = 1 - x^3$. What is the value of the function when

a) $x = 0$. **$y = 1$** b) $x = -1$. **$y = 2$**

c) $x = q$. **$y = 1 - q^3$** d) $x = -q$. **$y = 1 + q^3$**

Problem Let $f(x) = 4x^2$. Write what results when f operates on each argument.

a) $f(r)$ **$4r^2$** b) $f(t)$ **$4t^2$**

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c) $f(x^5)$ $4x^{10}$

d) $f(x-5)$ $4(x-5)^2 = 4x^2 - 40x + 100$

e) $f(1/x^2)$ $4/x^4$

f) $f(\frac{1}{2}\sqrt{x})$ x

Problem If $h(x) = -2$, then

a) $h(x^3) = -2$ b) $h(x+5) = -2$ c) $h(10) = -2$

The function h operates on every argument the same way. It produces -2.

h is called a constant function.

Problem Function of a function Let $f(x) = x^2$ and $g(x) = x + h$. Write the function

a) $f(g(x)) = f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$

b) $g(f(x)) = g(x^2) = x^2 + h$

Problem Let $f(x) = x - 3$, and $g(x) = 3 - x$. Write the functions $f(g(x))$ and $g(f(x))$.

$$f(g(x)) = f(3-x) = (3-x) - 3 = -x$$

$$g(f(x)) = g(x-3) = 3 - (x-3) = 3 - x + 3 = 6 - x$$

Problem Let $f(x) = x^5$ and $g(x) = x^{1/5}$. Write the functions $f(g(x))$ and $g(f(x))$.

$$f(g(x)) = f(x^{1/5}) = (x^{1/5})^5 = x^1 = x$$

$$g(f(x)) = g(x^5) = (x^5)^{1/5} = x$$

Problem This expression --

$$\frac{f(x+h) - f(x)}{h}$$

-- is called the Newton quotient or the difference quotient. Calculating and simplifying it is a fundamental task in differential calculus.

For each function $f(x)$, determine the difference quotient in a simplified form.

a) $f(x) = 2x + 1$

$$\frac{2(x+h)+1-(2x+1)}{h} = \frac{2x+2h+1-2x-1}{h}$$

$$= \frac{2h}{h}$$

$$= 2.$$

b) $f(x) = x^2$

1) $\dfrac{(x+h)^2 - x^2}{h} = \dfrac{x^2 + 2xh + h^2 - x^2}{h}$

2) $= \dfrac{2xh + h^2}{h}$

3) $= 2x + h.$

In Line 1) we squared the binomial $x + h$.
In Line 2) we subtracted the x^2's.
In Line 3) we divided both the numerator and denominator by h.

c) $f(x) = \frac{1}{x}$

$$1)\quad \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x-(x+h)}{(x+h)x}}{h}$$

$$2)\quad = \frac{x-x-h}{(x+h)x} \cdot \frac{1}{h}$$

$$3)\quad = \frac{-h}{(x+h)x} \cdot \frac{1}{h}$$

$$4)\quad = -\frac{1}{(x+h)x}$$

In Line 1) we added the fractions in the numerator of the complex fraction.

In Line 2) we removed the parentheses in the numerator, and multiplied by the reciprocal of the denominator.

In Line 3) we subtracted the x's.

In Line 4) we canceled the h's as -1, which on multiplication with 1 makes the fraction itself negative.

Let us start with an example:

Here we have the function **f(x) = 2x+3**, written as a flow diagram: The **Inverse Function** goes the other way: So the inverse of: 2x+3 is: (y-3)/2 The inverse is usually shown by putting a little "-1" after the function name, like this: $f^{-1}(y)$ *We say "**f inverse** of y"* So, the inverse of f(x) = 2x+3 is written: $f^{-1}(y) = (y-3)/2$

(I also used **y** instead of **x** to show that we are using a different value.)

The cool thing about the inverse is that it should give us back the original value:

Example:

Using the formulas from above, we can start with $x = 4$: $f(4) = 2\times4+3 = 11$

We can then use the inverse on the 11: $f^{-1}(11) = (11-3)/2 = 4$

And we magically get **4** back again!

We can write that in one line: $f^{-1}(f(4)) = 4$

"f inverse of f of 4 equals 4"

So applying a function f and then its inverse f^{-1} gives us the original value back again:

$$f^{-1}(f(x)) = x$$

We could also have put the functions in the other order and it still works:

$f(f^{-1}(x)) = x$

Example:

Start with: $f^{-1}(11) = (11-3)/2 = 4$

And then: $f(4) = 2\times4+3 = 11$

So we can say: $f(f^{-1}(11)) = 11$

Solve Using Algebra

We can work out the inverse using Algebra. **Put "y" for "f(x)"** and solve for x:

The function:	$f(x) = 2x+3$
Put "y" for "f(x)":	$y = 2x+3$
Subtract 3 from both sides:	$y-3 = 2x$
Divide both sides by 2:	$(y-3)/2 = x$
Swap sides:	$x = (y-3)/2$
Solution (put "$f^{-1}(y)$" for "x") :	$f^{-1}(y) = (y-3)/2$

This method works well for more difficult inverses.

Let us start with an example:

Here we have the function **f(x) = 2x+3**, written as a flow diagram: The **Inverse Function** goes the other way: So the inverse of: 2x+3 is: (y-3)/2 The inverse is usually shown by putting a little "-1" after the function name, like this: $f^{-1}(y)$ *We say "f inverse of y"* So, the inverse of $f(x) = 2x+3$ is written: $f^{-1}(y) = (y-3)/2$

(I also used **y** instead of **x** to show that we are using a different value.)

The cool thing about the inverse is that it should give us back the original value:

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with *y*. This is done to make the rest of the process easier.
2. Replace every *x* with a *y* and replace every *y* with an *x*.
3. Solve the equation from Step 2 for *y*. This is the step where mistakes are most often made so be careful with this step.
4. Replace *y* with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!

5. Verify your work by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

That's the process. Most of the steps are not all that bad but as mentioned in the process there are a couple of steps that we really need to be careful with.

In the verification step we technically really do need to check that both $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are true. For all the functions that we are going to be looking at in this section if one is true then the other will also be true. However, there are functions (they are far beyond the scope of this course however) for which it is possible for only of these to be true. This is brought up because in all the problems here we will be just checking one of them. We just need to always remember that technically we should check both.

Let's work some examples.

Example

Given $f(x) = 3x - 2$ find $f^{-1}(x)$ $f^{-1}(x)$.

Solution

Now, we already know what the inverse to this function is as we've already done some work with it. However, it would be nice to actually start with this since we know what we should get. This will work as a nice verification of the process.

So, let's get started. We'll first replace $f(x)$ with y.

$$y = 3x - 2$$

Next, replace all x's with y and all y's with x.

$$x = 3y - 2$$

Now, solve for y.

$$x + 2 - 3y$$

$$\frac{1}{3}(x+2) = y$$

$$\frac{x}{3} + \frac{2}{3} = y$$

Finally replace y with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

Now, we need to verify the results. We already took care of this in the previous section, however, we really should follow the process so we'll do that here. It doesn't matter which of the two that we check we just need to check one of them. This time we'll check that $(f \circ f^{-1})(x) = x$ is true.

$$\begin{aligned}(f\circ f^{-1})(x) &= f\left[f^{-1}(x)\right]\\ &= f\left[\frac{x}{3}+\frac{2}{3}\right]\\ &= 3\left(\frac{x}{3}+\frac{2}{3}\right)-2\\ &= x+2-2\\ &= x\end{aligned}$$

Example 2 Given $g(x)=\sqrt{x-3}$ find $g^{-1}(x)$, $x\geq 0$ $x\geq 0$.

Solution

Now the fact that we're now using $g(x)$ instead of $f(x)$ doesn't change how the process works. Here are the first few steps.

$$y=\sqrt{x-3}$$
$$x=\sqrt{y-3}$$

Now, to solve for *y* we will need to first square both sides and then proceed as normal.

$$\begin{aligned}x &= \sqrt{y-3}\\ x^2 &= y-3\\ x^2+3 &= y\end{aligned}$$

This inverse is then,

$$g^{-1}(x)=x^2+3$$

Finally let's verify and this time we'll use the other one just so we can say that we've gotten both down somewhere in an example.

$$\begin{aligned}(g^{-1}\circ g)(x) &= g^{-1}\left[g(x)\right]\\ &= g^{-1}\left(\sqrt{x-3}\right)\\ &= \left(\sqrt{x-3}\right)^2+3\\ &= x-3+3\\ &= x\end{aligned}$$

So, we did the work correctly and we do indeed have the inverse.

Before we move on we should also acknowledge the restrictions of $x\geq 0$ that we gave in the problem statement but never apparently did anything with. Note that this restriction is required to make sure that the inverse, $g^{-1}(x)$ given above is in fact one-to-one.

Without this restriction the inverse would not be one-to-one as is easily seen by a couple of quick evaluations.

$g^{-1}(1) = (1)^2 + 3 = 4$ $\qquad$ $g^{-1}(-1) = (-1)^2 + 3 = 4$

Therefore, the restriction is required in order to make sure the inverse is one-to-one.

The next example can be a little messy so be careful with the work here.

Example 3 Given $h(x) = \dfrac{x+4}{2x-5}$ find $h^{-1}(x)$.

Solution

The first couple of steps are pretty much the same as the previous examples so here they are,

$$y = \frac{x+4}{2x-5}$$
$$x = \frac{y+4}{2y-5}$$

Now, be careful with the solution step. With this kind of problem it is very easy to make a mistake here.

$$\begin{aligned} x(2y-5) &= y+4 \\ 2xy - 5x &= y+4 \\ 2xy - y &= 4+5x \\ (2x-1)y &= 4+5x \\ y &= \frac{4+5x}{2x-1} \end{aligned}$$

So, if we've done all of our work correctly the inverse should be,

$$h^{-1}(x) = \frac{4+5x}{2x-1}$$

Finally we'll need to do the verification. This is also a fairly messy process and it doesn't really matter which one we work with.

$$\begin{aligned} (h \circ h^{-1})(x) &= h\left[h^{-1}(x)\right] \\ &= h\left[\frac{4+5x}{2x-1}\right] \\ &= \frac{\frac{4+5x}{2x-1} + 4}{2\left(\frac{4+5x}{2x-1}\right) - 5} \end{aligned}$$

Okay, this is a mess. Let's simplify things up a little bit by multiplying the numerator and denominator by $2x-1$.

$$(h \circ h^{-1})(x) = \frac{2x-1}{2x-1} \cdot \frac{\frac{4+5x}{2x-1}+4}{2\left(\frac{4+5x}{2x-1}\right)-5}$$

$$= \frac{(2x-1)\left(\frac{4+5x}{2x-1}+4\right)}{(2x-1)\left(2\left(\frac{4+5x}{2x-1}\right)-5\right)}$$

$$= \frac{4+5x+4(2x-1)}{2(4+5x)-5(2x-1)}$$

$$= \frac{4+5x+8x-4}{8+10x-10x+5}$$

$$= \frac{13x}{13}$$

$$= x$$

Wow. That was a lot of work, but it all worked out in the end. We did all of our work correctly and we do in fact have the inverse.

EQUIVALENCE RELATIONS

Definition of an Equivalence Relations

A relation R on a set A is an equivalence relation if and only if R is

1. reflexive,
2. symmetric, and
3. transitive

Note: any equivalent relation must satisfy all three (3) conditions

Recall the definition of an equivalence relation. In general, equivalence relation results when we wish to 'identify' two elements of a set that share a common attribute. The definition is motivated by observing that any process of 'identification' must behave somewhat like the equality relation, and the equality relation satisfies the reflexive (x = x for all x), symmetric (x = y implies y = x), and transitive (x = y and y = z implies x = z) properties.

Example 1

Let R be the relation on the set R real numbers defined by xRy iff x − y is an integer. Prove that R is an equivalence relation on R.

Proof

1. Reflexive:

Suppose x ∈ R. Then x − x = 0, which is an integer. Thus, x R x.

2. Symmetric:

Suppose x, y ∈ R and x R y. Then x − y is an integer. Since y − x = −(x − y), y − x is also an integer. Thus, y R x.

3. **transitive**

Suppose x, y and z ∈ R, x R y and y R z. Then x − y and y − z are integers. Thus, the sum (x − y) + (y − z) = x − z is also an integer, and so x R z. Thus, R is an equivalence relation on R.

Discussion

Let R be the relation on the set of real numbers R in Example 1. Prove that if xRx' and yRy' , then (x + y)R(x'+ y').

Proof.

Suppose xRx' and yRy' . In order to show that (x+y)R(x '+y '), we must show that (x + y) − (x ' + y ') is an integer. Since (x + y) − (x ' + y ') = (x − x ') + (y − y '), and since each of x−x ' and y −y ' is an integer (by definition of R), (x−x ') + (y −y ') is an integer. Thus, (x + y)R(x ' + y ').

Exercise

In the example above, show that it is possible to have xRx' and yRy',

but (xy)R̸(x' y').

Let V be the set of vertices of a simple graph G. Define a relation R on V by vRw iff v is adjacent to w. Prove or disprove: R is an equivalence relation on V.

PROPERTIES OF INTERGERS

Mathematical equations have their own manipulative principles. These principles or properties help us to solve such equations. The properties of integers are the basic principle of the mathematical system and it will be used throughout the life. Hence, it's very essential to understand how to apply each of them to solve math problems. Basically, there are three properties which outline the backbone of mathematics. They are:

- Associative property
- Commutative property
- Distributive property

Properties Of Integers

Property	Operations on Integers			
Name	Addition	Subtraction	Multiplication	Division*
Closure	a + b ∈ Z	a - b ∈ Z	a × b ∈ Z	a ÷ b ∉ Z
Commutative	a + b = b + a	a - b ≠ b - a	a × b = b × a	a ÷ b ≠ b ÷ a
Associative	(a + b) + c = a + (b + c)	(a - b) - c ≠ a - (b - c)	(a × b) × c = a × (b × c)	(a ÷ b) ÷ c ≠ a ÷ (b ÷ c)
Distributive	a × (b + c) = ab + ac	a × (b - c) = ab - ac	Not applicable	Not applicable

where a, b, c ∈ Z

*b is a non-zero integer

All properties and identities for addition and multiplication of whole numbers are applicable to integers also. Integers include the set of positive numbers, zero and negative numbers which c… represented with the letter Z.

$Z = \{.........-5,-4,-3,-2,-1,0,1,2,3,4,5..................-5,-4,-3,-2,-1,0,1,2,3,4,5........\}$

Some of the Properties of Integers are given below:

Property 1: Closure property

Among the various properties of integers, closure property under addition and subtraction states that the sum or difference of any two integers will always be an integer i.e. if x and y are any two integers, $x + y$ and $x - y$ will also be an integer.

Ex: $3 - 4 = 3 + (-4) = -1 - 1$;

$(-5) + 8 = 3$.

The results are integers.

Closure property under multiplication states that the product of any two integers will be an integer i.e. if x and y are any two integers, xyxy will also be an integer.

Ex: $9 \times 6 = 54$; $(-5) \times (3) = (-5) \times (3) = -15$, which are integers.

Division of integers doesn't follow the closure property, i.e. the quotient of any two integers x and y, may or may not be an integer.

Ex : $(-3) \div (-6) = \frac{1}{2}$, is not an integer.

Property 2: Commutative property

Commutative property of addition and multiplication states that the order of terms doesn't matter, result will be same. Whether it is addition or multiplication, swapping of terms will not change the sum or product. Suppose, xx and yy are any two integers, then

$\Rightarrow x + y = y + x \Rightarrow x \times y = y \times x$

Ex: $4 + (-6) = -2 = (-6) + 4$;

$10 \times (-3) = -30 = (-3) \times 10$

But, subtraction ($x - y \neq y - x$) and division ($x \div y \neq y \div x$) are not commutative for integers and whole numbers.

Ex: $4 - (-6) = 10$; $(-6) - 4 = -10$

$\Rightarrow 4 - (-6) \neq (-6) - 4$

Ex: $10 \div 2 = 5$; $2 \div 10 = 1/5$

$\Rightarrow 10 \div 2 \neq 2 \div 10$

Property 3: Associative property

Associative property of addition and multiplication states that the way of grouping of numbers doesn't matter; the result will be same. One can group numbers in any way but the answer will remain same. Parenthesis can be done irrespective of the order of terms. Let x, y and z be any three integers, then

$x + (y + z) = (x + y) + z$

$\Rightarrow x \times (y \times z) = (x \times y) \times z$

Ex: $1 + (2(-3)) = 0 = (1 + 2) + (-3)$;

$1 \times (2 \times (-3)) = -6 = (1 \times 2) \times (-3)$

Subtraction of integers is not associative in nature i.e. $x - (y - z) \neq (x - y) - z$.

Ex: $1 - (2 - (-3)) = -4$; $(1 - 2) - (-3) = 2$; $1 - (2 - (-3)) \neq (1 - 2) - (-3)$

Property 4: Distributive property

Distributive property explains the distributing ability of an operation over another mathematical operation within a bracket. It can be either distributive property of multiplication over addition or distributive property of multiplication over subtraction. Here integers are added or subtracted first and then multiplied or multiply first with each number within the bracket and then added or subtracted. This can be represented for any integers x, y and z as:

$\Rightarrow x \times (y + z) = x \times y + x \times z$

$\Rightarrow x \times (y - z) = x \times y - x \times z$

Ex: $-5(2 + 1) = -15 = (-5 \times 2) + (-5 \times 1)$

Property 5: Identity Property

Among the various properties of integers, additive identity property states that when any integer is added to zero it will give the same number. Zero is called additive identity. For any integer x,

$x + 0 = x = 0 + x$

Multiplicative identity property for integers says that whenever a number is multiplied by the number 1 it will give the integer itself as the product. Therefore, the integer 1 is called the multiplicative identity for a number. For any integer x, $x \times 1 = x = 1 \times x$

If any integer multiplied by 0, product will be zero: $x \times 0 = 0 \times x$

If any integer is multiplied by -1, product will be opposite of the number:

$x \times (-1) = -x = (-1) \times x$

Property	**Addition**	**Multiplication**
Commutative	$x + y = y + x$	$x \times y = y \times x$
Associative	$x + (y + z) = (x + y) + z$	$x \times (y \times z) = (x \times y) \times z$
Distributive	$x \times (y + z) = x \times y + x \times z$	
Identity	$x + 0 = x = 0 + x$	$x * 1 = x = 1 * x$

EXX

UNIT THREE
SEQUENCE AND SERIES

Objectives

By the end of this unit, you should be able to

1. Write down the next three terms of a given sequence
2. Distinguish between sequence and series
3. Distinguish between linear and exponential sequences
4. Find the n^{th} term and the sum of the first n terms of an arithmetic progression.
5. Find the n^{th} term and the sum of the first n terms of a geometric progression.
6. Determine the terms of a recursively defined sequences.
7. Define a sequence using summation notation.

Now Read on:

A sequence is a set of numbers written in a definite order. We call each number a term of sequence.
A **series** on the other hand is a sequence writhen with operational sign separating the various terms.
A series is formed by the sum of the terms of a sequence.
i.e. when the terms of a sequence is added a series is formed.

Example of sequence

i) 2, 4, 6, 8 …….. ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} + \cdots$ ....

Examples of series

i) $2 + 4 + 6 + 8 =$ …….. ii) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

A sequence is therefore a set of terms in a defined order with a rule for obtaining each term. The first term of the sequence is denoted 'a' or 'U'.

A finite sequence contains only finite or a definite number of terms. Thus a sequence whose terms can be counted. e.g. (a) 1, 2, 3, -,-,-, 90 and (b) 2, 4, 6, 8, 10

An infinite sequence on the other hand is an unending sequence i.e.a sequence whose first or last term is not known. e.g. (a) 1, 2, 3, 4, -,-,- (b) -, -, -, 2, 4, 6, 8, 10

Example:

Write down the next three terms of each of the following sequence:

a) {1, 3, 5, 7, ……….}
b) {1, 3, 6, 10………..}
c) $\{-12, -6, -3, -\frac{2}{3} \ldots \ldots\}$
d) {1, 2, 4, 8, ……….}
e) $0, -\frac{1}{2}, -1, -\frac{3}{2}$

Solution:

a) Each term is +2, hence the next three terms are: 9, 11, and 13
Thus {1, 3,5, 7, 9, 11, 13]

b) 1   3   6   10,   10 + 5,   15 + 6   21 + 7
   2   3   4   15   21   28

$\Rightarrow$ the next three terms are: 15, 21, and 28
Hence, the sequence is {1, 3, 6, 10, 15, 21, and 28}

c) $-12, -6, -3\ -\frac{3}{2}$ it can be seen that each term is divided by 2, (÷ 2)

$-\frac{12}{2} = -6, \quad -\frac{6}{2} = -3, \quad -\frac{3}{2} = -\frac{3}{2}$ $\Rightarrow$ the next three terms are:

$\frac{-3}{2} \div 2 = \frac{-3}{4}, \quad \frac{-3}{4} \div 2 = \frac{-3}{8}, \ \& \ \frac{-3}{16}$

Thus $\left\{-12, -6, -3, \frac{-3}{2}, \frac{-3}{4}, \frac{-3}{8}, \frac{-3}{16}\right\}$

d) 1, 2, 4, 8, -,-,-, it can be seen that each term is double the preceding term
i.e. $1 + 1 = 2,\ 2 + 2 = 4,\ 4 + 4 = 8,\ 8 + 8 = 16,\ 16 + 16 = 32$
$\therefore$ the sequence {1, 2, 4, 8, 16, 32, 64}

e) The next three terms are: $2, -\frac{5}{2}, -3$

LINEAR SEQUENCE OR ARITHMETIC PROGRESS (AP)

A sequence in which successive terms increase (or decrease) by a constant is called a linear sequence or an arithmetic progression. In other words, an AP is one in which each term is derived from the previous term by adding a constant, **d**. The constant can either be positive or negative and it's called **common difference.**

Thus if the n terms of an AP are $U_1, U_2, U_3, U_4 \ldots\ldots U_n$, then

$U_1 = a$, $U_2 = a + d$, $U_3 = a + 2d$, $U_4 = a + 3d$, etc.

i.e

1^{st}	2^{nd}	3^{rd}	4^{th}	n^{th}
U_1	U_2	U_3	U_4	U_n
↓	↓	↓	↓	↓
a	a + d	a + 2d	a + 3d	a + (n – 1)d

In general, an AP with first term, '**a**' and common difference, **d**, has an n^{th} term of

$$U_n = a + (n - 1)\,d$$

The common difference is found by subtracting a term from the term that precedes it.

i.e. $U_4 - U_3 = U_2 - U_1 = d$

E.g. Write down the 5^{th} and the n^{th} term of the AP in

(i) 1, 5, -, -, -, ii) 2, $1\frac{1}{2}$, –, –, –, (iii) First term is 5 and common difference is 3.

(iv) Find the 15^{th} term of the sequence -3, 2, 7, - , -, -,

Solution

i) $a = 1$, $d = 5 - 1 = 4$

$\Rightarrow U_5 = a + 4d$

$= 1 + 4(4) = 17$

Or

$U_5 = a = (n - 1)d$

$= 1 + (5 - 1)\,4 = 17$

The n^{th} term $U_n = a + (n + m10\,d$

$= 1 + (n - 1)\,4$

$= 1 + 4n - 4$

$\therefore U_n = 4n - 3$

ii) $2, \frac{1}{2} \Rightarrow a = 2$, and $d = \frac{1}{2} - 2 = \frac{-1}{2}$

$\therefore U_5 = a + 4d$

$= 2 + 4\left(-\frac{1}{2}\right) = 2 - 2 = 0$

$\Rightarrow U_5 = 0.$

The n^{th} term (general term), $U_n = a + (n - 1)d$

$= 2 + (n + 1)\left(-\frac{1}{2}\right)$

$= 2 - \frac{1}{2}n + \frac{1}{2}$

$= \frac{5}{2} + \frac{1}{2}n$ factorise

$\Rightarrow U_n = \frac{1}{2}(5 - n)$

iii) $a = 5$, and $d = 3$

Using $U_5 = a + 4d$

$= 5 + 4(3) = 17$

The general term $U_n = 5 + (n - 1)3$

$= 5 + 3n - 3 = 3n + 2$

iv) $a = -3$, $d = 2 - (-3) = 5$

$U_n = a + (n - 1)\,d$

$\rightarrow U_{15} = -3 + (15 - 1)(5)$

$= -3 + 70$

$\Rightarrow U_{15} = 67$

More examples

1) The 8th term of an AP is twice the 5th term. If the firs term is 4, find the common difference and the 15th term. List the first 5 terms of the sequence.

Solution

8th term = 2 (5th term)

$\Rightarrow a + 7d = 2(a + 4d)$

$a + 7d = 2a + 8d$

$\Rightarrow a = -d \quad \therefore d = -4.$

Hence, $U_{15} = a + (n - 1)d$

$= 4 + (15 - 1)(-4)$

$= -52$

The first five terms of the sequence are: 4, 0, −4, −8, −12

Series

$2 + 4 + 6 + 8 + 10 + 12 \ldots$

$2S_n = 2a + (n-1)d$

$\frac{2S_n}{2} = \frac{n}{2}[2a + (n-1)d]$

2) The 3rd term of an AP is −1 and the 10th term is 20. Find the

i) first term and the common difference
ii) 20th term.
iii) general term of the sequence

Solution

(i) Given that 3rd term = -1 ⇒ $a + 2d = -1$ ………(1)

Also the 10th term = 20 ⇒ $a + 9d = 20$ ……….(2)

Solving the two equations gives $a = -7$ and $d = 3$

Hence, the first term is −7 and the common difference is 3

(ii) Using $U_n = a + (n - 1)d$

$\Rightarrow U_{20} = -7 + (20 - 1)(3) = 50$

(iii) $U_n = a + (n - 1)d$

$U_{20} = 77 + (n - 1)(3)$

$= -7 + 3n - 3$

$\Rightarrow U_n = 3n + 10.$

The sequence is -7, -4, -1, 2, 5 …………. d = 3, a = 7

3) A 5th term of an AP is thrice the 2nd term. If the first term is 8, find

a) The common difference
b) The 11th term of the sequence
c) The 10the term of the sequence.

Answers

a) $a = 8, d = 2a = 16$
b) $U_n = 16_n - 18$
c) $U_{10} = 16(10) - 8 = 152$

The Sum of the first *n* terms of an AP

Consider the sum of the first nth term of the AP with the following terms;

$a, \ a + d, \ a + 2d, \ a + 3d, -, -, -, \ a + (n - 1)d$

⇒the sum, $S_n = a + (a + d) + (a + 2d) + (a + 3d) + l$ …………..(1)

where l is the last term

Reversing equation (1), we have

$S_n = l + (l - d) + (l - 2d) + (a + 2d) + (a + d) + a$ ………..(2)

Adding equation (1) and (2), we have

$2Sn = 2a + (n - 1)d + 2a + (n - 1)d, -, -, -, n$ times

$\Rightarrow 2S_n = (2a + (n - 1)d)n$

$\Rightarrow S_n = \frac{(2a + (n-1)d)n}{2}$

Hence the sum of the first n terms of an AP is

$$Sn = \frac{n}{2}[2a + (n - 1)d]$$

$S_n = \frac{n}{2}[a + l] \qquad l = a + (n-1)d$

But $a + (n - 1)d = l$

$\Rightarrow S_n = \frac{n}{2}[a + l]$

Note: For an AP, given S_n and S_{n-1}

$$U_n = S_n - S_{n-1}$$

E.g. The sum of the first 5 terms of a sequence is 30 and the sum of the first 4 terms is 20. Find the 5th term.

Solution

Given $S_5 = 30$, and $S_4 = 20$,

$U_n = S_n - S_{n-1}$

$\Rightarrow U_5 = S_5 - S_4$

$\Rightarrow U_5 = 30 - 20 = 10$

Examples:

1. Find the sum of the firs 10 terms of the sequence terms

a) -7, -4, -1, 2............

b) 0, - ½ , -1, $\frac{-3}{2}$

c) 2, $1\frac{1}{2}$, ...

Solution

a) $a = 7$, $d = 3$

$Sn = \frac{n}{2}[2a + (n-1)d]$

$\Rightarrow 5_{10} = \frac{10}{2}[2(-7) + (10-1)3]$

$= 5(-14 + 27) = 65$

b) $a = 0$, and $d = -\frac{1}{2}$

$S_{10} = \frac{10}{2}\left[2(0) + (10-1)\left(\frac{-1}{2}\right)\right]$

$= 5\left(0 - \frac{9}{2}\right) = -\frac{45}{2} = -22.5$

c) $a = 2$, and $d = -$ ½

$U_{10} = \frac{10}{2}\left[p2(2) + (10-1)\left(-\frac{1}{2}\right)\right]$

$= 5\left(4 - \frac{9}{2}\right) = \frac{-5}{2}$

$= -2.5$

2. In an AP the 6th term is thrice the 2nd term and the 9th term is 27. Find

a) the first term and the common difference

b) sum of the 1st n terms

c) the sum of the 1st 20 terms

Solution

(a) $U_6 = 3(U_2)$

$a + 5d = 3(a + d)$

$a + 5d = 3a + 3d \Rightarrow a = d$ ....(1)

Also $U_9 = 27$

$\Rightarrow a + 8d = 27$ ....(2)

Put (1) into (2) gives

$d + 8d = 27 \Rightarrow d = 3$ and $a = 3$ (since $a = d$)

Hence, the First term = 3 and the Common difference = 3

b) $S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{n}{2}[2(3) + (n-1)3]$

$= \frac{n}{2}(6 + 3n - 3) = \frac{n}{2}(3n + 3)$

Hence, the sum of the first n terms is $S_n = \frac{3n}{2}(n + 1)$

c) $S_{20} = \frac{3(20)}{2}(20 + 1)$

= 30 (21) = 630.

Hence the sum of the 1st 20 terms is 630.

3) In an anathematic progression, the 8th term is twice the 4th term and the 20th term is 40. Find

i) The common difference

ii) The sum of the 1st 10 terms

iii) The sum of the terms from the 8th to the 20th inclusive.

Solution

$U_8 = 2(U_4)$ and given that $U_{20} = 40$

$\Rightarrow a + 7d = 2(a + 3d)$

$\Rightarrow a + d7d = 2a + 6d$

$\Rightarrow a = d$ ....(1)

Also $a + 19d = 40$ ....(2)

Substituting (1) into (2), gives

$d + 19d = 40 \Rightarrow 20d = 40$

$\therefore d = 2$

GP = 1

i) The common difference is 2

ii) The sum of the 1st 10 tems is S_{10}

$S_n = \frac{n}{2}[\,2a + (n-1)d\,]$

$S_{10} = \frac{10}{2}[2(2) + (10-1)2]$

$= 5(4 + 18) = 110.$

iii) The sum of 8th to 20th inclusive is given by

$S_{20} - S_7$

But $S_{20} = \frac{20}{2}[\,2(2) + (20-1)(2)] = 420$

And $S_7 = \frac{7}{2}[2(2) + (7-1)(2)] = 56$

$\therefore$ The sum from 8th to 20th terms inclusive is $420 - 56 = 364$.

4) In an AP, the sum of the 2nd and the 5th term is 25. If the sum of 5th and 8th term is 43, find

a) The first term and the common difference

b) the sum of the 1st 10 terms

c) the 20th term of the sequence

Solution

Given that $U_2 + U_5 = 25$

$\Rightarrow 2a + 5d = 25$ ....(1)

Also, given that $U_5 + U_8 = 43$

$\Rightarrow 2a + 11d = 45$ ....(2)

Solving (1) and (2) simultaneously

$\Rightarrow d = 3$ and $a = 5$

Hence the first term is 5 and the common difference is 3

b) $S_{10} = \frac{10}{2}[\,2(5) + (10-1)(3)] = 185.$

c) $U_{20} = a + (n-1)d$

$= 5 + (20-1)(3) = 62.$

Exercise

1. The 5th term of a linear sequence is 12 and the 12th term is 25. Find the
   a) First term
   b) Common difference
   c) The sum of the 1st 30 terms.
   d) Sum of the terms from 5th to 15th inclusive.

Application Questions

✓ 1) Mr. Asare Salary start at GH¢24,000 and increased by annual increment of GH¢36,000. After how many years is the maximum salary reached? How many years would Mr. Asare earn altogether, after 10 years in the job?

Solution

The first term, a = 24,000, common difference, d = 2000 and U_n = 36000. Substituting these into U_n = a (n + 1) d

⇒ 36000 = 24000 + (n − 1) 2000

⇒ $n = 7 years$

Hence, Mr. Asare will reach his maximum salary in 7 years time.

Since the sequence breaks down after 7 years, from the 8th year he will receive a fixed amount of GH¢36000 for 3 years.

So the amount the 3 years ⇒ 3 x 36,000 = 108,000

And for the first 7 years, $S_7 = \frac{7}{2}[2(24{,}000) + (6)2000] = 210{,}000$

∴ For ten years = 210,000 + 108,000 = GH¢318,000.00

2) The starting salary for a job is GH¢7,000.00 per annum and increases by GH¢500.00 per annum at the end of each year. Calculate the total salary earned in ten years.

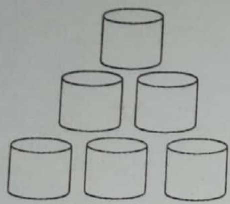
Solution

a = 7000   d, = 500   n = 10

⇒ $S_{10} = 5[2(7000) + 9 \times 500]$

= GH¢92,500

3) A tea seller is arranging some tins of milo in triangular pile, as shown below. If she wants the pile to be 12 rows high, how many milo tin must she use?



Solution

The sequence is = 1, 2, 3, 4……………….

⇒ a = 1, d = 1 n = 1

$S_{12} = \frac{12}{2}[2(1) + (12 - 1)1]$

= 6 (2 + 11) = 78 milo tins.

4) During the 1st second of a journey, a car travels 2m. it travels 4m during the 2nd seconds, 6m during the 3rd seconds and so on. Assuming that the distance continues to increase, find:

a) The distance travelled in the 10th second.

b) The total distance travelled up to time 10s.

Solution

1st	2nd	3rd ……………….
↓	↓	↓
$2m$	$4m$	$6m$…………………

⇒ a = $2m$,   d = $2m$

a) $U_{10} = 2 + (10 - 1)\,2 = 20m$.

b) $S_{10} = \frac{10}{2}[2(2) + (10 - 1)2] = 110m$.

NB. *The sum of an arithmetic series is equal to the product of the number of terms and half the sum of the two extreme terms.*

e.g. Find the sum of the sequence; 2 , 4, 6, 8, 10

$S_5 = 5 \times \frac{1}{2}(2 + 10) = 30$

5) Mr. M otey's rend increased by GH¢60eveyr year. If in 20 years, he paid a total of GH¢21,400 as rent, find

i) His rent for the first year.

ii) His rent in the 20th hear.

Solution

Let a = rent for the first year
S_{20} = GH¢21400, n = 20, and d = 60
But $S_n = \frac{n}{2}[2a + (n-1)d]$
$\Rightarrow 214000 = \frac{20}{2}[\,2a + (20-1)60]$
21400 = 10 (2a + 1140) ⇒ a = 500
Hence, the rent for the 1st year = Gh¢500.00
ii) $U_n = a + (n-1)\,d$
$\Rightarrow U_{20} = 500 + (20-1)\,60 = 1640.$
∴ The rent in 20th year = GH¢1,640.00

EXPONENTIAL SEQUENCE OR GEOMETRIC PROGRESSION (GP)

A sequence in which each term is a constant multiple of the preceding term, is called a Geometric Progression or an exponential sequence. The constant multiple is called the common ratio (r). E.g. 2, 4, 8, 16, -, -, -, each term is twice the preceding term.
Thus 2, 2(2), 4(2), 8(2). The constant 2 is called the common ratio.
If U_1, U_2, U_3, U_4,-, -, -, are the 1st, 2nd, 3rd and 4th terms respectively of a sequence then the common ratio, $r = \frac{U_2}{U_1} = \frac{U_3}{U_2} = \frac{U_4}{U_3}$

The terms of a geometric progression
In general if a GP has a first term '**a**' and a common ratio '**r**', then the first n terms are

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4 \ldots\ldots\ldots\ldots ar^{n-1}$$

∴ The nth term of a GP,

$$U_n = ar^{n-1}$$

Examples

1) Find the 15th term of the GP 3, 6, 12.........

Solution
$a = 3,\ r = \frac{6}{3} = \frac{12}{6} = 2$
But $U_n = ar^{n-r}$
$\Rightarrow U_{15} = 3(2)^{15-1} = 3(2)^{14}$

2) Find the common ratio of a GP whose first term is 36 and whose 4th term is $4\frac{1}{2}$. Hence the nth term of the sequence.

Solution
$a = 36,\quad U_4 = 4\frac{1}{2}$
$\Rightarrow ar^3 = 4\frac{1}{2}$
But $a = 36 \Rightarrow 36r^3 = 4\frac{1}{2}$
$\Rightarrow r^3 = \frac{1}{8} \Rightarrow r = \frac{1}{2}$
$U_n = ar^{n-1}$
$= 36\left(\frac{1}{2}\right)^{n-1} = 36\left(\frac{1}{2}\right)^{n} \times 2$
$= 72\left(\frac{1}{2}\right)^{n}$

3) Find the number of terms in the exponential sequence $1, \frac{1}{3}, \frac{1}{9}, \ldots, \ldots, \ldots, \frac{1}{243}$

Solution
$a = 1,\quad r = \frac{1}{3}$ and $U_n = \frac{1}{243}$
Using $U_n = ar^{n-1}$
$\Rightarrow \frac{1}{243} = 1\left(\frac{1}{3}\right)^{n-1}$
$\Rightarrow \frac{1}{243} = \left(\frac{1}{3}\right)^{n-1}$

$\Rightarrow \frac{1}{3^5} = \left(\frac{1}{3}\right)^{n-1} \Rightarrow \left(\frac{1}{3}\right)^5 = \left(\frac{1}{3}\right)^{n-1}$

$\Rightarrow n - 1 = 5 \qquad \therefore n = 6.$

Hence there are 6 terms in the sequence.

4) In a GP, the sum of the 2nd and the 3rd terms is 6 and the sum of the 3rd and the fourth term is 12. Find the first term and the common ratio.

Solution

$U_2 + U_3 = 6$ and $U_3 + U_4 = 12$

$\Rightarrow ar + ar^2 = 6$ ……………….(1)

$ar^2 + ar^3 = -12$ ………………..(2)

Divide equation (2) by (1)

$\frac{a^r + ar^3}{ar + ar^2} = \frac{-12}{6}$

$\frac{a^r + ar^3}{ar + ar^2} = \frac{-12}{6}$

Putt $r = -2$ into any one of the equations gives 'a' = 3.

5) In a GP, the 3rd term is 5 and the 6th term is 135. Find
a) The common ratio.
b) The 12th term of the sequence.

Solution

$U_6 = 5 \Rightarrow ar^2 = 5$……………(1)

Also $U_6 = 135 \Rightarrow ar^5 = 135$…………(2)

(1) ÷ (2)

$\Rightarrow \frac{ar^5}{ar^2} = \frac{135}{5}$

$\Rightarrow r^3 = 27$

$\therefore r = 3.$

b) $U_{12} = ar^{n-1}$

Put $r = 3$ into equation (1) $\Rightarrow a = \frac{5}{9}$

$\Rightarrow U_{12} = \frac{5}{9}(3)^{12-1} = \frac{5}{9}(3)^{11}$ ro 3,936.6

The sum of the first n terms of GP

The first n term of a an exponential sequence are; a, ar, ar^2, ar^3, - , -, -, ar^{n-1}

Let Sn represent the sum of the sequence. Thus

$S_n = a + arar^3 + ar^4 + \ldots\ldots + ar^{n-1}$ …………..(1)

Multiply equation (1)by r, we have

$rS_n = ar + ar^2 + ar^3 + ar^5 \ldots\ldots + ar^n$.................(2)

Now, equation (1) – (2)

$S_n - rS_n = a - ar^n$

$\Rightarrow S_n = \frac{a(r-r^2)}{1-r}, r < 1$

Alternatively, (2) – (1) we, have

$rS_n - S_n = ar^n + a$

$\Rightarrow (r-1)S_n = a(r^n - r)$

$\therefore S_n = \frac{a(r^n - 1)}{r-1}, \; r > 1$

The sum to infinity is given by

$S\infty = \frac{a}{1-r}$

Example:

1) Find the sum of the first five term of the sequence 1, 3, 9, 27………..

Solution

a = 11, r = 3, Since r > 1, use

$S_n = \frac{a(r^n - 1)}{r-1}$

$\Rightarrow S_5 = \frac{1[(3)^5-1]}{3-1} = \frac{240}{2} = 121.$

2) The 7th term of a GP is 56 and the 4th term is 7. Find the sum of the first 10 terms of the sequence.

Solution

$U_7 = 56 \Rightarrow ar^6 = 56$................(1)

Also $U_4 = 7 \Rightarrow ar^3 = 7$..............(2)

(1) ÷ (2)

$\Rightarrow \frac{ar^6}{ar^3} = \frac{56}{7} \Rightarrow r^3 = 8 \therefore r = 2.$

$S_n = \frac{1}{81}\frac{(3^{10}-1)}{3-1} = \frac{1}{81}\frac{(59}{2}$

Put r = 2 into (2) $ar^3 = 7$

$\Rightarrow a(2)^3 = 7$

$8a = 7 \Rightarrow a = \frac{7}{8}$

$\therefore$ Using $S_n = \frac{a(r^n-1)}{r-1}$

$\Rightarrow S_{10} = \frac{7}{8}\frac{[2^{10}-1]}{2-1} = \frac{7}{8}(1023).$

= 895. 125.

3) The 5th term of a GP is 162 and the 8th term is 4374. Find the

a) 10th term.

b) Sum of the first 10 terms

Answers: r = 3, and a = 2

a) $U_{10} = 39{,}366$

b) $S_{10} = 59{,}048$

4) Write down the sum of the first n terms of the following series

i) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \ldots\ldots\ldots\ldots$

ii) $12 + 6 + 3 + 1\frac{1}{3} + \cdots \ldots\ldots$

iii) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots \ldots$

Solution

i) $S_n = \frac{a(r^n-1)}{r-1}$

$= \frac{1\left(1-\left(\frac{1}{3}\right)^n\right)}{1-r} = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$

$= S\infty = \frac{3}{2}(1) = \frac{3}{2}$

ii) $S_n \frac{1\left(1-\left(\frac{1}{3}\right)^n\right)}{2/3} = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^n\right)$

$S\infty = \frac{2}{3}(1) = \frac{2}{3}$

5) If the sum of infinity of a GP is three times the first term. What is the common ratio?

Solution

$S\infty = \frac{1}{1-r}$ But $S\infty = 3a$

$\Rightarrow \frac{a}{1-r} = 3a$ Crosing multiply

$\Rightarrow a = 3a(1-r)$ dividing by

$\Rightarrow 1 = 3(1 - r) \Rightarrow 1 = 3 - 3r$

$\Rightarrow 3r = 2$

$\therefore r = \frac{2}{3}$

6) The sum of n terms of a certain series is $4^n - 1$ for all values of n. Find the first three terms and the n^{th} term and show that the series is a GP

Solution

$S_n = 4^n - 1$

Using the relation $U_n = S_n - S_{n-1}$, we have

$U_n = 4^n - 1 - (4^{n-1} - 1)$

$= 4^n - 1 - 4^{n-1} + 1$

$= 4^{n-1}(4-1)$

$\Rightarrow U_n = 3 \times 4^{n-1}$

$U_1 = 3 \times 4^0 = 3$
$U_2 = 3 \times 4 = 12$
$U^3 = 3 \times 4^2 = 48.$

∴the first three terms of the GP are 3, 12, and 48.

Find the series to be a GP;

$\Rightarrow \frac{U_1}{U_1} = \frac{U_3}{U_2}$

$\Rightarrow \frac{12}{3} = \frac{48}{12} = 4$

Since $\frac{U_3}{U_1}$ is equal to $\frac{U_3}{U_2}$ it follows that the series is a GP

NB: Start with the $U_n = 4^n-1$, the terms are 3, 15, 63

Application Questions on GP

1) A factor produced 250 tonnes of plastic goods in 2003. Since then, it increased its production by 20 percent every year. Calculate the total production of the factory from 2003 to 2008.

Solution

2003	2004	2005…………
↓	↓	↓
250	300	360 …………

This is a GP of first term a = 250, and common ratio, $r = \frac{6}{5}$

Substituting into the formula $S_n = \frac{a\,(r^2-1)}{(r-1)0}$, gives the sum, S_6 = 2,482.48 tonnes.

2) A mother bought 100kg of rice for the family. Each week, she cooked one-tenth of the rice left over from the previous week. Find the
   i) total quantity of the rice cooked by the end of the 10^{th} week.
   ii) How many weeks she took to cook 90kg of rice.

Solution

1^{st} Week	2^{nd} Week	3^{rd} Week	4^{th} Week……….
↓	↓	↓	↓
10	9	$\frac{81}{10}$	$\frac{729}{100}$

$\Rightarrow a = 10$ and $r = \frac{9}{10}$

i) $S_n = \frac{10-(1-\left(\frac{9}{10}\right)^{10})}{1-\frac{9}{10}}$

$= 65.132kg$

ii) $90 = \frac{10-(1-\left(\frac{9}{10}\right)^{9})}{1-\frac{9}{10}}$

$= \left(\frac{9}{10}\right)^n = \frac{1}{10}$

$Log_{0.9}(0.1) = 21.55 = 22$ weeks.

3) A Car depreciates by 20% each year. The cost of the car when bought is GH¢30,000. Calculate the
   a) value of the car after the third year.
   b) value of the car after the fifth year.

Ans. a) 19,200 b) 12,288

Self-Assessment Questions

1) Find the 11^{th} term of the sequences with the first $1\frac{1}{2}$ and common difference $\frac{-7}{4}$.
2) Find the 12^{th} term of the sequence 8, $7\frac{1}{4}$, $6\frac{1}{2}$, … …
3) Find the number of terms in the sequence
i) 2, 7, 12, …………………42

ii) 5, 2, -1, ………………, -58

4) The 5^{th} and 10^{th} terms of a linear sequence are -12 and -27 respectively. Find the sequence and its 15^{th} term.

5) The sum of three consecutive integers is 33. Determine the numbers.

6) The consecutive odd numbers p, q and r are such that the sum of p and r is 34. Find p, q and r.

7) The sum consecutive terms of an AP have sum 15 and product 80. Find the numbers.

8) Find the sum of the first thirteen terms of the linear sequence whose 8^{th} term is 18 and 11^{th}

9) Find the sum of all integers between 8 and 90.

10) The set P is a subset of the set of integers. If P = P {multiples of 4 less than 101}, find the sum of all the element of P.

11) Find the number of terms of the AP $4 + 6\frac{1}{2} + 9 + 11\frac{1}{2} + \ldots\ldots$ Needed to make a total of 26.

12) If S_n is the sum of the first n terms of the sequence $1, (1+x), (1+x)^2 \ldots\ldots, (1+x)^{n-1}$, show that

$$S_n = n + \frac{1}{2n}(n-1)x + \frac{1}{6}(n-1)(n-2)x^2$$

Neglecting all terms in x^3 and higher powers of x.

b) If n = 20 and $x = 0.1$, Calculate the approximate value of S_n.

14) In an experimental sequence, the 6^{th} term is 8 times the 3^{rd} term and the sum of the 7^{th} and 8^{th} terms is 192. Find the sum of the 5^{th} to 11^{th} terms inclusive.

15) The first, third and ninth terms of a linear sequence are the first three terms of an experimental sequence. If the seventh term of the linear sequence is 14, calculate.

i) the 20^{th} term of the linear sequence

ii) the sum of the first twelve terms of the exponential sequence

16) A man starts saving on 1^{st} April. He saves 1pound the first day 2pounds the second day, 4pounds the third day, and so on. Doubling the amount every day. If he managed to keep on saving under this system this system until the end of the month (30 days), how much would he save? Give your answer in pounds, correct to three significance figures.

Answers

1) -16. (2) $-\frac{1}{4}$ (3) (i) 9 terms. (ii) 22

(4) The sequence is 0, -3, -6, -9…………… The 15^{th} term is -42

(5) The numbers are 10, 11 and 12. (6) p, q and r are: 15, 17 and 19.

(7) 2, 5 and 8 (8) 208 (9) 3969 (10) 1300 (11) 9

(13) b) 50.4 (14) 2032 (15) i) 40 ii) 531,440 (16) 10,700,000 (3sgf)

RECURRENT RELATIONS

A second way of defining a sequence is to assign a value to the first (or the first few) term(s) and specify the *n*th term by a formula or equation that involves or more of the terms preceding it. Sequences defined this way are said to be defined recursively, and the rule of formula is called a **recursive formula**.

Example 1. Write down the first five terms of the following recursively defined sequence.

$$S_1 = 1 \qquad S_n = nS_{n-1}$$

The fist term is given as $S_1 = 1$. To get the second term, we use $n = 2$. the formular $S_n = ns_{n-1}$ to get $S_2 = 2s_1 = 2.1 = 1$. To get the third term, we use n = 3 in the value of the preceding temr. The first five term are

$S_1 = 1$

$S_2 = 2.1 = 2$

$S_3 = 3.2 = 6$

$S_4 = 4.3 = 12$

$S_5 = 5.24 = 120.$

Do you recognize this sequence? $S_n = n!$

Example 2. Write down the first five terms of the following recursively defined sequence.

Solution

We are given the first two terms. To get the third term requires that we know both of t he previous two terms. That is,

$u_1 = 1$

$u_2 = 1$

$u_3 = u_1 + u_2 = 1 + 1 = 2$

$u_4 = u_2 + u_3 = 1 + 2 = 3$

$u_5 = u_3 + u_4 = 2 + 3 = 5$

The sequence defined in Example 2 is called the **Fibonacci sequence**, and the terms of this sequence are called **Fibonacci numbers**.

Example 3. A sequence of number U, U_2, U_3..........U_n.... satisfies the relation $U_{n+1} + n^2 = nU_n + 2$, for all integers $n \geq 1$. If $U_1 = 2$, find

a) the values of U_2, U_3 and U_4.

b) the expression for U_n in terms of n

c) the sum of the first n terms of the sequence.

Solution

Given $U_n + N^2 = nU_n + 2$; $n \geq 1, U = 2$

When n = 1, $U_2 + 1 = U_1 + 2$ but $U_1 = 2$

$\Rightarrow U_2k + 1 = 2 + 2$

$\therefore U_2 = 3.$

When n = 2, $U_3 = 4 = 2U_2 + 2$

$\Rightarrow U_3 = 2U_2 - 2$ but $U_2 = 3$

$\Rightarrow U_3 = 2(3) - 2 = 4$

When n = 3, $U_4 + 9 = 3U_3 + 2$

$U_4 = 3U_3 - 7$ but $U_{3=4}$

$\Rightarrow U_4 = 3(4) - 7 = 5$

Hence$U_1, U_2, U_3, U_4 = 2, 3, 4, 5$..............

b) $U_n = a + (n - 1)d$

$= 2 + (n - 1) = n + 1$

d) $S_n = \frac{n}{2}(2a + (n -= 1)d] = \frac{n}{2}(3 + n)$

Example 4. A sequence of numbers U_1, U_2, U_3 .... Satisfies the relation $(3n - 2)U_{n+1} = (3n + 1)U_n$ for all positive integers n. If $U_1 = 1$, find

a) U_3 and U_4

b) the expression for U_n

c) the expression for S_n

Solution

When n=1, $(3 - 2)U_2(3 + 1)_1$

When n = 1, $(3 - 2)_2 = (3 + 1)U_1$

$U_2 = 4U_1$ but $U_1 = 1$.

$\Rightarrow U_2 = 4.$

When n = 2, $4U_{3=}7U_2$

$\Rightarrow U_3 = 6$ etc.

Similarly $U_4 = 10$

b) $U_n = a + (n-1)d$

$= 1 + (n-1)3 = 3x - 2$

c) $S_n = \frac{n}{2}(3x - 1)$

RECURRENT DECIMALS

Recurrent decimals are repeated decimals. Every recurrent decimal represents a rational number. For example

$\frac{1}{3} = 0.33333$..................or $0.\dot{3}$

$\frac{2}{1} = 0.181818$ .... or $0.18\ \dot{1}\dot{8}$

Every recurrent decimal can be expressed as quotient $\left(\frac{r}{q}\right)$ of two integers by first writing it as an infinite exponential series, with first term a, and common ratio, r. For example.

(a) $\frac{1}{3} = 0.3333 \Rightarrow 0.3 + 0.3 + 0.003 + \cdots \ldots\ldots$

$= \frac{3}{10} + \frac{3}{10^2} + \frac{18}{100^3} + \ldots\ldots$

(b) $\frac{8}{15} = 0.54333 = 0.5 + 0.03 + 0.003 + \cdots$

$= \frac{5}{10} + \frac{3}{10^2} + \frac{3}{10^2} + \cdots \ldots\ldots\ldots\ldots\ldots$

Examples

1) Express $0.\dot{6}$ as an infinite geometric series and hence find the sun of the series.

Solution

$0.6 = 0.6 = 0.06 + 0.006 + \ldots\ldots\ldots\ldots$

$= \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \ldots\ldots$

Hence 0.6 is an infinite series of first term $\frac{6}{10}$ as a common ratio $\frac{1}{10}$

ii) Since $|r| < 1$, the sum to infinite

$\Rightarrow 0.6 = \frac{1}{1-r} = \frac{\frac{6}{10}}{1-\frac{1}{10}} = \frac{2}{3}$

2) Express the recurring decimal $0.\ddot{2}\dot{1}$ in the form $\frac{p}{q}$, when p and q are integers.

Solution

$0.\ddot{2}\dot{1} = 0.21212121 = 0.21 + 0.0021 + 0.000021 + 0.21 + 0.0021 + 0.0000021 +$ ....

$= \frac{21}{100} + \frac{21}{100^2} + \frac{21}{100^3} + \ldots\ldots$

This is an infinite series with first term $\frac{21}{100}$ and common ratio $r = \frac{1}{100}$

Hence $0.\ddot{2}\dot{1} = \frac{a}{1-r}$

$= \frac{\frac{21}{100}}{1-\frac{1}{100}} = \frac{7}{33}$

3) Express $0.1\dot{6}$ 0.16 in the form $\frac{p}{q}$, where p and q are integers.

Solution

$0.1\dot{6} = 0.16666$

$= 0.1 + 0.06 + 0.006 + 0.00006 + \ldots\ldots$

$= \frac{1}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \ldots\ldots\ldots$

But $\frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \cdots \ldots\ldots$ *is a GP* with first term $\frac{6}{10^2}$ and common ratior $= \frac{1}{10}$

Thus $\frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \ldots\ldots = \frac{\frac{6}{100}}{1-\frac{1}{10}} = \frac{1}{15}$

$\therefore 0.1\dot{6} = \frac{1}{10} + \frac{1}{15} = \frac{1}{6}$

Self-Assessment Questions

Express the following recurring decimals in the form $\frac{p}{p}$

i) $0.1\dot{3}\dot{6}$ ii) $0.41\dot{6}$ iii) $0.4\dot{1}\dot{3}$

Ans: (i) $\frac{3}{22}$ $ii)\ \frac{5}{12}$ $iii)\ \frac{409}{990}$

ARITHMETIC, GEOMETRIC AND HARMONIC MEANS

(a) ***Arithmetic Mean***

If three numbers **a, b, c** are consecutive terms of an AP, then **b** is called, the *arithmetic mean* of '**a**' and '**c**'. Thus if **a, b, c** are in AP then the common difference is therefore b – a or c – b

$\Rightarrow$ b – a = c – b

$\Rightarrow$ 2b = c + a

$\therefore$ b = $\frac{c+a}{2}$

$\therefore$ The arithmetic mean of **a** and **c** is $\frac{a+c}{2}$. Thus average of '**a**' and '**c**'

For example, Find the arithmetic mean of 4 and 64.

Solution:

Arithmetic Mean = $\frac{4+64}{2} = \frac{68}{2} = 34.$

b) ***Geometric Mean***

If **a, b, c** are consecutive terms of a GP, then **b** is called the **geometric mean** of **a** and C

The common ratio, r is $\frac{b}{a}$ *or* $\frac{c}{b}$

$\Rightarrow \frac{b}{a} = \frac{c}{b}$

$\Rightarrow b^2 = ac$

$\therefore$ b = $\sqrt{ac}$

For example, Find the geometric mean of 4 and 64

Solution

Geometric Mean = $\sqrt{(4 \times 64}$

= $\sqrt{256}$ = 16.

c) ***Harmonic Mean***

The reciprocal of the harmonic mean of two numbers is the arithmetic mean of their reciprocals. i.e. If '**a**' and '**c**' are two numbers then

$$\frac{1}{Harmonic\ Mean} = \frac{\frac{1}{a}+\frac{1}{c}}{2}$$

$$\Rightarrow \frac{1}{HN} = \frac{c+a}{2ac}$$

$$\Rightarrow \frac{1}{HM} = \frac{a+c}{2ac}.$$

$$\therefore HM = \frac{2ac}{a+c}.$$

For example, Find the harmonic mean of 5 and 20

Solution

$$\frac{1}{HM} = \frac{\frac{1}{5}+\frac{1}{20}}{2} = \frac{5}{40} = \frac{1}{8}$$

$\Rightarrow$ HM = 8

NB: AM > GM > HM. Arithmetic mean is the *greatest*

SUMMATION NOTATION

It is often important to be able to find the sum of the first n terms of a sequence $\{a_n\}$, that is

$$a_1 + a_2 + a_3 + \cdots \ldots\ldots\ldots. + a_n$$

Rather than write down all the terms, we introduce a more concise way to express the sum, called **summation notation**. Using summation notation we write the sum as

$$a_1 + a_2 + a_3 + \ldots\ldots + a_n = \sum_{k=1}^{n} a_k$$

$\frac{1}{2}n(n-1)$

The symbol $\sum$ called sigma (Greek letter) is simply an instruction to sum, or add up, the terms. The integer k is called the index of the sum, it tells you where to start the sum and where to end it. The expression

$$\sum_{(k=1)}^{n} k!$$

is an is an instruction to add the terms a_k of the sequence $\{a_n\}$ starting with $k = 1$ *and* ending with $k = n$. Wer read the expressing ad "the sum of a_k from $k = 1$ *to* $k = n$."

for example, Write out each sum.

$$a)\ \sum_{k=1}^{n} \frac{1}{k} \qquad b)\ \sum_{k=1}^{n} k!$$

Solution

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n} \qquad b)\ \sum_{k-1}^{n} k! = 1! + 2! + \cdots \ldots + n!$$

Examples

Express each sum using summation notation

a) $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + 9^2$ b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots + \frac{1}{2^{n-1}}$

Solution

a) The sum $1^2 + 2^2 + 3^3 + \ldots\ldots + 9^2$ has 9 terms, eachof the form k^2, and strt at k = 1 and ends at k = 9.

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + 9^2 = \sum_{k=1}^{9} k^2$$

b) The sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}}$ has n times, each of the form $\frac{1}{2^{n-1}}$ $\sum_{k=1}^{n} \frac{1}{2^{k-1}}$

The index of summation need not always begins at 1 or end in n; for example, we could have express the sum in expanded (b) as

$$\sum_{k=0}^{n-a} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} \ldots\ldots\ldots + \frac{1}{2^{n-1}}$$

each represents the same sum as the one given in example above

Properties of Sequence

If $\{a_n\}$ and $\{b_n\}$ are two sequence and c is real number, then:

$\sum_{k=1}^{n}(ca_k) = ca_1 + ca_2 + \ldots\ldots\ldots + ca_n = c(a_1 + a_2 + \ldots + a_n) = c\sum_{k=1}^{n} a_k$

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j}$, where $0 < j < n$.

Formulas for Sums of Sequences

$$\sum_{k=1}^{n} c = c + c + \cdots \ldots\ldots\ldots + c = cn \quad c \text{ is a real number}$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots \ldots\ldots\ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots \ldots\ldots \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \left[\frac{n(n+1)}{2}\right]$$

Examples

1) Find the sum of each of the following sequence

a) $\sum_{k=1}^{5}(3k)$ b) $\sum_{k=1}^{10}(k^2 + 1)$ c) $\sum_{k=1}^{24}(k^2 - 7k + 2)$

d) $\sum_{k=6}^{20}(4k^2)$

Solution

a) $\sum_{k=1}^{5}(3k) = 3\sum_{k=1}^{5} k$

$= 3\left(\frac{5(5+1)}{2}\right)$

$= 3(15)$

$= 45$

$$b)\ \sum_{k=1}^{10}(k^3) + \sum_{k=1}^{10} 1$$

$$= \left(\frac{10(10+1)}{2}\right)^2 + 1(10)$$

$$= 3025 + 10$$

$$= 3035.$$

$$c)\ \sum_{k=1}^{24}(k^2 - 7k + 2) = \sum_{k=1}^{24} k^2 - \sum_{k=1}^{24} 2$$

$$= \sum_{k=1}^{24} k^2 - 7\sum_{k=1}^{24} k + \sum_{k=1}^{24} 2$$

$$= \frac{24(24+1)(2.24+1)}{6} - 7\left(\frac{24(24+1)}{2}\right) + 2(24)$$

$$= 4900 - 2100 + 48$$

$$= 2484$$

d) Notice that the index of summation start at 6. We use property (4) as follows

$$\sum_{k=6}^{20}(4k^2) = 4\sum_{k=6}^{20} k^2 = 4[\sum_{k=1}^{20} k^2 - \sum_{k-1}^{5} k^2] = 4\left|\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6}\right|$$

$$= 4[2870 - 55] = 11,260.$$

2) Find the sum S_n of the first n terms of the sequence $\{a_n\} = \{3n + 5\}$; that is, find

$8 + 11 + 14 + \ldots\ldots\ldots\ldots + (3n + 5) = \sum_{k-1}^{n}(3k + 5)$

Solution

The sequence $\{a_n\} = \{3n + 5\}$ is an arithmetic sequence withfirst term $a_1 = 8$ and the nth term $a_n = 3n + 5$. To find the sum S_n, we use formula (4) above

$S_n = \sum_{k=1}^{n}(3k + 5) = \frac{n}{2}[8 + 5)] = \frac{n}{2}(3n + 13)$

Unit 4
Intuitive Treatment of Convergence and Divergence of Series

Convergence Tests

- *Geometric Series*: A geometric series is the series of the form $\sum_{n=0}^{\infty} ar^n$. The series converges if $|r|<1$ and has the sum, $S_\infty = \frac{a}{1-r}$; where a is the first term of the series and r is the common ratio. The series diverges if $|r| \geq 1$.
- *p-Series*: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$

Examples:

1. Determine if $\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+\ldots\ldots\ldots\ldots+\frac{6}{10^n}$ converges or diverges and if it converges, find its sum.

   Solution:

$$a=\frac{6}{10}, \qquad r=\frac{6/100}{6/10}=\frac{1}{10}$$

Since $|r|<1$, then the series converges and has the sum;

$$S_\infty = \frac{a}{1-r} = \frac{\frac{6}{10}}{1-\frac{1}{10}}$$

$$= \frac{6/10}{9/10}$$

$$= \frac{6}{9}$$

$$= \frac{2}{3}$$

Sum is $\frac{2}{3}$

2. Determine whether the series given below converges or diverges. Find the limit if it converges.

$$2+\frac{2}{3}+\frac{2}{3^2}+\ldots\ldots+\frac{2}{3^{n-1}}$$

Solution:

$$a=2, \qquad r=\frac{2/3}{2}=\frac{1}{3}$$

Since $r = \frac{1}{3} < 1$, the series converges

$$S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}}$$

$$= \frac{2}{\frac{2}{3}}$$

$$= 3$$

3. Determine whether $\sum_{n=0}^{\infty} 3^n = 1+3+9+27+81+....$ converges and find the sum

Solution:

$$\sum_{n=0}^{\infty} 3^n = 1+3+9+27+81+....$$

This is a GP of first term, $a=1$ and $r=3$

Since $r = 3 > 1$, the series diverges

4. Determine whether $\sum_{n=0}^{\infty} \frac{2}{3^n} = \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, ...$ converges and find the sum

Solution:

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, ...$$

This is a GP of first term, $a = 2$ (since $n = 0$)

and $r = \frac{\frac{2}{3}}{2} = \frac{1}{3}$

Since $r = \frac{1}{3} < 1$, the series converges.

NB: The sequence in Example 4 is the same as the sequence put in a different way in Example 2

Self Assessment

Determine whether the following series converges or diverges

i. $\sum_{n=1}^{\infty} \frac{1+3n^2}{1+n^2}$

ii. $\sum_{1}^{\infty} \frac{n}{n^2+1}$

Corollary (The N[th]- Term Test)

(i) If a series $\sum_{n=1}^{\infty} a_n$, is convergent, then

$$\lim_{n\to\infty} a_n = 0$$

(ii) If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(n^{th} - term test for divergence)

(iii) If $\lim_{n\to\infty} a_n = 0$,

then the $\sum_{n=1}^{\infty} a_n$ series may either converge or diverge.

NB: The nth Term Test for Divergence is an intrinsic test that can be used for all series without restriction.

Examples:

a. Determine whether the series $\sum_{n=1}^{\infty} \frac{1+3n^2}{1+n^2}$ converges or diverges.

Solution:

Recall that if $\lim_{n\to\infty} a_n \neq 0$, then the series is certainly divergent.

$$\lim_{n\to\infty}\left(\frac{1+3n^2}{1+n^2}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n^2}+\frac{3n^2}{n^2}}{\frac{1}{n^2}+\frac{n^2}{n^2}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{n^2}+3}{\frac{1}{n^2}+1}\right)$$

$$= 3$$

Since the limiting value is not zero, it implies that $\sum_{n=1}^{\infty} \frac{1+3n^2}{1+n^2}$ diverges.

b. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ converges or diverges.

Solution:

Recall that if $\lim_{n\to\infty} a_n \neq 0$, then the series is certainly divergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{n/n}{2n/n + 1/n}$$

$$= \lim_{n\to\infty} \frac{1}{2 + 1/n}$$

$$= \frac{1}{2+0}$$

$$= \frac{1}{2}$$

Since $\lim_{n\to\infty} a_n \neq 0$, the series is divergent.

c. Determine whether the series $\sum_{n=1}^{\infty}\left(\frac{n^2+n+3}{2n^2+1}\right)$ converges or diverges.

Solution:

$$\lim_{n\to\infty}\left(\frac{n^2+n+3}{2n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}+\frac{3}{n^2}}{2+\frac{1}{n^2}}\right)$$

$$= \frac{1}{2}$$

Since $\frac{1}{2} \neq 0$, $\left(i.e.\ \lim_{n\to\infty} a_n \neq 0\right)$, the series is divergent.

d. Determine whether the sequence $a_n = \frac{1}{4n^2}$ converges or diverges.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{4n^2}$$

$$= \frac{1}{4(\infty)}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Since $\lim_{n\to\infty} a_n = 0$, we know nothing: the series can either converge or diverge.

NB: We use the fact that the sequence is in the form $\frac{1}{n^p}$ and thus converges since $p > 1$

Self Assessment

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges or diverges.

2. Determine whether $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges or diverges.

Theorem

A series in which the sum (S_n) of n terms of the series tend to a definite numerical value, as $n \to \infty$, is called convergent series. That is a series converges when the $\lim_{n\to\infty} S_n$ gives you a definite numerical value. If S_n does not tend to a definite value as $n \to \infty$, the series is said to be divergent.

Examples:

1. Determine whether the series $1+3+9+27+81+....$ converges or diverges

   Solution:

   $1+3+9+27+81+.... = \sum_{n=0}^{\infty} 3^n$

   This is a GP of first term, $a = 1$ and $r = 3$

$$\therefore S_n = \frac{a(1-r^n)}{1-r} = \frac{1(1-3^n)}{1-3}$$

$$= \frac{1-3^n}{-2}$$

$$= \frac{3^n-1}{2}$$

Sn = a / (r-1)

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3^n-1}{2}$

$$= \infty$$

Since the limit of S_n is not a definite numerical value, it implies the series diverges.

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$ converges or diverges

Solution:

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = S_n$$

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{10^n}$

in determinate

$$= \frac{3}{\infty}$$

$$= 0$$

Since the limit of S_n is a definite numerical value, that is 0, the series converges.

3. Determine whether $\sum_{1}^{\infty} \frac{5n+3}{2n-7}$ converges or diverges

Solution:

$$S_n = \sum_{1}^{\infty} \frac{5n+3}{2n-7}$$

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{1}^{\infty} \left(\frac{5n+3}{2n-7}\right)$

$$= \lim_{n\to\infty} \left(\frac{5n+3}{2n-7}\right)$$

$$= \lim_{n\to\infty}\left(\frac{5n/n + 3/n}{2n/n - 7/n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{5 + 3/n}{2 - 7/n}\right)$$

$$= \frac{5+0}{2-0}$$

$$= \frac{5}{2}$$

Since the limit of S_n is a definite numerical value, that is $\frac{5}{2}$, the series converges.

Self Assessment

Determine whether $\sum_1^\infty \frac{2n^2+4n-3}{5n^2-6n+1}$ converges or diverges

Unit 4 Intuitive Treatment of Convergence and Divergence of Series

Convergence Tests

- *Geometric Series*: A geometric series is the series of the form $\sum_{n=0}^{\infty} ar^n$. The series converges if $|r|<1$ and has the sum, $S_\infty = \frac{a}{1-r}$; where a is the first term of the series and r is the common ratio. The series diverges if $|r|>1$.
- *p-Series*: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p<1$

Examples:

5. Determine if $\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+............+\frac{6}{10^n}$ converges or diverges and if it converges, find its sum.

Solution:

$$a=\frac{6}{10}, \qquad r=\frac{6/100}{6/10}=\frac{1}{10}$$

Since $|r|<1$, then the series converges and has the sum;

$$S_\infty = \frac{a}{1-r} = \frac{\frac{6}{10}}{1-\frac{1}{10}}$$

$$= \frac{6/10}{9/10}$$

$$= \frac{6}{9}$$

$$= \frac{2}{3}$$

6. Determine whether the series given below converges or diverges. Find the limit if it converges.

$$2+\frac{2}{3}+\frac{2}{3^2}+......+\frac{2}{3^{n-1}}$$

Solution:

$$a=2, \qquad r=\frac{2/3}{2}=\frac{1}{3}$$

Since $r = \frac{1}{3} < 1$, the series converges

$$S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}}$$

$$= \frac{2}{\frac{2}{3}}$$

$$= 3$$

7. Determine whether $\sum_{n=0}^{\infty} 3^n = 1+3+9+27+81+....$ converges and find the sum

Solution:

$\sum_{n=0}^{\infty} 3^n = 1+3+9+27+81+....$

This is a GP of first term, $a = 1$ and $r = 3$

Since $r = 3 > 1$, the series diverges

8. Determine whether $\sum_{n=0}^{\infty} \frac{2}{3^n} = \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, ...$ converges and find the sum

Solution:

$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, ...$

This is a GP of first term, $a = 2$ (since $n = 0$)

and $r = \frac{\frac{2}{3}}{2} = \frac{1}{3}$

Since $r = \frac{1}{3} < 1$, the series converges.

NB: The sequence in Example 4 is the same as the sequence put in a different way in Example 2

Self Assessment

Determine whether the following series converges or diverges

iii. $\sum_{n=1}^{\infty} \frac{1+3n^2}{1+n^2}$

iv. $\sum_{1}^{\infty} \frac{n}{n^2+1}$

Corollary (The N^th^- Term Test)

(iv) If a series $\sum_{n=1}^{\infty} a_n$, is convergent, then

$$\lim_{n\to\infty} a_n = 0$$

(v) If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
(n^{th} - term test for divergence)

(vi) If $\lim_{n\to\infty} a_n = 0$,

then the $\sum_{n=1}^{\infty} a_n$ series may either converge or diverge.

NB: The Nth Term Test for Divergence is an intrinsic test that can be used for all series without restriction.

Examples:

e. Determine whether the series $\sum_{n=1}^{\infty} \frac{1+3n^2}{1+n^2}$ converges or diverges.

Solution:

Recall that if $\lim_{n\to\infty} a_n \neq 0$, then the series is certainly divergent.

$$\lim_{n\to\infty}\left(\frac{1+3n^2}{1+n^2}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n^2}+\frac{3n^2}{n^2}}{\frac{1}{n^2}+\frac{n^2}{n^2}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{n^2}+3}{\frac{1}{n^2}+1}\right)$$

$$= 3$$

Since the limiting value is not zero, it implies that $\sum_{n=1}^{\infty} \frac{1+3n^2}{1+n^2}$ diverges.

f. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ converges or diverges.

Solution:

Recall that if $\lim_{n\to\infty} a_n \neq 0$, then the series is certainly divergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{n/n}{2n/n + 1/n}$$

$$= \lim_{n\to\infty} \frac{1}{2 + 1/n}$$

$$= \frac{1}{2+0}$$

$$= \frac{1}{2}$$

Since $\lim_{n\to\infty} a_n \neq 0$, the series is divergent.

g. Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{n^2+n+3}{2n^2+1} \right)$ converges or diverges.

Solution:

$$\lim_{n\to\infty} \left(\frac{n^2+n+3}{2n^2+1} \right) = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}+\frac{3}{n^3}}{2+\frac{1}{n^2}} \right)$$

$$= \frac{1}{2}$$

Since $\frac{1}{2} \neq 0$, $\left(i.e.\ \lim_{n\to\infty} a_n \neq 0 \right)$, the series is divergent.

h. Determine whether the sequence $a_n = \frac{1}{4n^2}$ converges or diverges.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{4n^2}$$

$$= \frac{1}{4(\infty)}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Since $\lim_{n\to\infty} a_n = 0$, we know nothing: the series can either converge or diverge.

NB: We use the fact that the sequence is in the form $\frac{1}{n^p}$ and thus converges since $p > 1$

Self Assessment

3. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges or diverges.

4. Determine whether $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges or diverges.

Theorem

A series in which the sum (S_n) of n terms of the series tend to a definite numerical value, as $n \to \infty$, is called convergent series. That is a series converges when the $\lim_{n\to\infty} S_n$ gives you a definite numerical value. If S_n does not tend to a definite value as $n \to \infty$, the series is said to be divergent.

Examples:

4. Determine whether the series $1+3+9+27+81+....$ converges or diverges

   Solution:

   $1+3+9+27+81+.... = \sum_{n=0}^{\infty} 3^n$

   This is a GP of first term, $a = 1$ and $r = 3$

$$\therefore S_n = \frac{a(1-r^n)}{1-r} = \frac{1(1-3^n)}{1-3}$$

$$= \frac{1-3^n}{-2}$$

$$= \frac{3^n-1}{2}$$

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3^n-1}{2}$

$$= \infty$$

Since the limit of S_n is not a definite numerical value, it implies the series diverges.

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$ converges or diverges

Solution:

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = S_n$$

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{10^n}$

$$= \frac{3}{\infty}$$

$$= 0$$

Since the limit of S_n is a definite numerical value, that is 0, the series converges.

6. Determine whether $\sum_{1}^{\infty} \frac{5n+3}{2n-7}$ converges or diverges

Solution:

$$S_n = \sum_{1}^{\infty} \frac{5n+3}{2n-7}$$

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{1}^{\infty} \left(\frac{5n+3}{2n-7}\right)$

$$= \lim_{n\to\infty} \left(\frac{5n+3}{2n-7}\right)$$

$$= \lim_{n \to \infty} \left(\frac{5n/n + 3/n}{2n/n - 7/n} \right)$$

$$= \lim_{n \to \infty} \left(\frac{5 + 3/n}{2 - 7/n} \right)$$

$$= \frac{5+0}{2-0}$$

$$= \frac{5}{2}$$

Since the limit of S_n is a definite numerical value, that is $\frac{5}{2}$, the series converges.

Self Assessment

Determine whether $\sum_{1}^{\infty} \frac{2n^2 + 4n - 3}{5n^2 - 6n + 1}$ converges or diverges

Unit 5: Comparison of Ratio and Root test

Ratio test

The ratio test looks at the ratio of a general term of a series to the immediately preceding term. The ratio test works by looking only at the nature of the series you're trying to figure out (as opposed to the tests which compare the test you're investigating to a known, benchmark series). If, in the limit, this ratio is less than 1, the series converges; if it's more than 1 (this includes infinity), the series diverges; and if it equals 1, you learn nothing, and must try a different test (Thus, not able to tell whether divergent or convergent). The ration test works especially well involving factorials as n! is in the power, like4^n.

Keep in mind that the *factorial* symbol (!) tells you to multiply like this:

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1.$

Notice how things cancel when you have factorials in the numerator and denominator of a fraction:

$$\frac{6!}{5!} = \frac{6.5.4.3.2.1}{5.4.3.2.1} = 6$$

*$(n+1)! = (n+1)n!$

$$\frac{5!}{6!} = \frac{5.4.3.2.1}{6.5.4.3.2.1} = \frac{1}{6}$$

In both cases, everything cancels but the 6.

$$\frac{(n+1)!}{n!} = n+1$$

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}$$

In the same way everything cancels but the $(n + 1)$. Last, it seems weird, but $0! = 1$.

Definition

Suppose we have the series$\sum a_n$. Define,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

NOTE: Notice that in the case of $L = 1$ the ratio test is pretty much worthless and we would need to resort to a different test to determine the convergence of the series.

Also, the absolute value bars in the definition of L are absolutely required. If they are not there it will be impossible for us to get the correct answer.

Let's take a look at some examples.

Try this one:

Does

$$\sum_{n=0}^{\infty} \frac{3^n}{n!}$$

converge or diverge?

Here's what you do. You look at the limit of the ratio of the $(n+1)$st term to the nth term:

$$\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{3^{n+1} \cdot n!}{(n+1)! \cdot 3^n}$$

$$= \lim_{n\to\infty} \frac{3}{n+1}$$

$$= \frac{3}{\infty + 1}$$

$$= 0$$

Because this limit is less than 1, $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges

$\sum_{n=0}^{\infty} \frac{3^n}{n!}$

Let $a_n = \frac{3^n}{n!}$

put $n+1$ for n

$a_{n+1} = \frac{3^{n+1}}{(n+1)!}$

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$\lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)!} \div \frac{3^n}{n!} \right|$

$\lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right|$

$L = \lim_{n\to\infty} \left| \frac{3^n \cdot 3!}{(n+1)n!} \cdot \frac{n!}{3^n} \right|$

$\lim_{n\to\infty} \cdot \left(\frac{3}{n+1} \right)$

$= \frac{3}{\infty+1}$

$= \frac{3}{\infty}$

$= 0$

Here's another series:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

What's your guess — does it converge or diverge? Look at the limit of the ratio:

What's your guess — does it converge or diverge? Look at the limit of the ratio:

Q

$\lim_{n\to\infty} \left| 1 + \frac{1}{n} \right| = e$

$\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

Soln

Let $a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}$

$a_{n+1} = \frac{(-10)^{n+1}}{4^{2(n+1)+1}(n+1+1)}$

$a_{n+1} = \frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$= \lim_{n\to\infty} \left| \frac{\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}}{\frac{(-10)^n}{4^{2n+1}(n+1)}} \right|$

$= \lim_{n\to\infty} \left| \frac{-10^{n+1}}{4^{2n+3}(n+2)} \cdot \frac{4^{2n+1}(n+1)}{-10^n} \right|$

$= \lim_{n\to\infty} \left| \frac{-10^n \cdot 10}{4^{2n} \cdot 4^3 (n+2)} \cdot \frac{4^{2n} \cdot 4 (n+1)}{-10^n} \right|$

$\lim_{n\to\infty} \left| \frac{-10(n+1)}{4^2(n+2)} \right| = \frac{10}{16} \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right|$

$= \frac{10}{16} \lim_{n\to\infty} \left| \frac{\frac{n}{n} + \frac{1}{n}}{\frac{n}{n} + \frac{2}{n}} \right|$

$= \frac{10}{16} \lim_{n\to\infty} \left| \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} \right| = \frac{10}{16} \lim_{n\to\infty} \left| \frac{1+0}{1+0} \right|$

$= \frac{10}{16}$

Because the limit is less than 1, $\sum_{n=1}^{\infty} \frac{-10}{4^{2n+1}(n+1)}$ Converges.

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1} \cdot n!}{(n+1)! \cdot n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1) \cdot n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$= e$ $\left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \text{ is one of those limits you need to memorize.} \right)$

≈ 2.718

Because the limit is greater than 1, $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

Determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Solution

With this first example let's be a little careful and make sure that we have everything down correctly. Here are the series terms a_n.

Let

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Recall that to compute a_{n+1} all that we need to do is substitute $n+1$ for all the n's in a_n.

$$a_{n+1} = \frac{(-10)^{n+1}}{4^{2(n+1)+1}((n+1)+1)} = \frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$$

Now, to define L we will use,

$$L = \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right|$$

$$L - \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}\frac{4^{2n+1}(n+1)}{(-10)^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-10(n+1)}{4^2(n+2)}\right|$$

$$= \frac{10}{16}\lim_{n\to\infty}\frac{n+1}{n+2}$$

$$= \frac{10}{16} < 1$$

So, $L < 1$ and so by the Ratio Test the series converges absolutely and hence will converge.

Determin e if the following series is convergent or divergent

$$\sum_{n=0}^{\infty}\frac{n!}{5^n}$$

Solution

Now that we've worked one in detail we won't go into quite the detail with the rest of these. Here is the limit.

$$L = \lim_{n\to\infty}\left|\frac{(n-1)!}{5^{n+1}}\frac{5^n}{n!}\right| = \lim_{n\to\infty}\frac{(n+1)!}{5\ n!}$$

In order to do this limit we will need to eliminate the factorials. We simply can't do the limit with the factorials in it. To eliminate the factorials we will recall from our discussion on factorials above that we can always "strip out" terms from a factorial. If we do that with the numerator (in this case because it's the larger of the two) we get,

$$L = \lim_{n\to\infty}\frac{(n+1)\ n!}{5\ n!}$$

at which point we can cancel the $n!$ for the numerator an denominator to get,

$$L = \lim_{n\to\infty}\frac{(n+1)}{5} = \infty > 1$$

So, by the Ratio Test this series diverges.

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{9^n}{(-2)^{n+1}n}$$

Solution

Comparison Test

$$L = \lim_{n\to\infty}\left|\frac{9^{n+1}}{(-2)^{n+2}(n+1)}\,\frac{(-2)^{n+1}\,n}{9^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{9\,n}{(-2)(n+1)}\right|$$

$$= \frac{9}{2}\lim_{n\to\infty}\frac{n}{n+1}$$

$$= \frac{9}{2} > 1$$

Therefore, by the Ratio Test this series is divergent.

In the previous example the absolute value bars were required to get the correct answer. If we hadn't used them we would have gotten $L = -\frac{9}{2} < 1$ which would have implied a convergent series!

Now, let's take a look at a couple of examples to see what happens when we get $L = 1$. Recall that the ratio test will not tell us anything about the convergence of these series. In both of these examples we will first verify that we get $L = 1$ $L = 1$ and then use other tests to determine the convergence.

Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$$

Solution

Let's first get L.

$$L = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)^2+1}\,\frac{n^2+1}{(-1)^n}\right| = \lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1} = 1$$

So, as implied earlier we get $L = 1$ which means the ratio test is no good for determining the convergence of this series. We will need to resort to another test for this series. This series is an alternating series and so let's check the two conditions from that test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n^2+1} = 0$$

$$b_n = \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} = b_{n+1}$$

The two conditions are met and so by the Alternating Series Test this series is convergent. We'll leave it to you to verify this series is also absolutely convergent.

Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty}\frac{n+2}{2n+7}$$

Comparison Test

Solution
Here's the limit.

$$L = \lim_{n\to\infty}\left|\frac{n+3}{2(n+1)+7}\,\frac{2n+7}{n+2}\right| = \lim_{n\to\infty}\frac{(n+3)(2n+7)}{(2n+9)(n+2)} = 1$$

Again, the ratio test tells us nothing here. We can however, quickly use the divergence test on this. In fact that probably should have been our first choice on this one anyway.

$$\lim_{n\to\infty}\frac{n+2}{2n+7} = \frac{1}{2} \neq 0$$

$$\lim_{n\to\infty}\frac{n+2}{2n+7} = \frac{1}{2} \neq 0$$

By the Divergence Test this series is divergent.

So, as we saw in the previous two examples if we get $L = 1$ from the ratio test the series can be either convergent or divergent.

There is one more thing that we should note about the ratio test before we move onto the next section. The last series was a polynomial divided by a polynomial and we saw that we got $L = 1$ from the ratio test. This will always happen with rational expression involving only polynomials or polynomials under radicals. So, in the future it isn't even worth it to try the ratio test on these kinds of problems since we now know that we will get $L = 1$.

Also, in the second to last example we saw an example of an alternating series in which the positive term was a rational expression involving polynomials and again we will always get $L = 1$ in these cases.

Let's close the section out with a proof of the Ratio Test.

Root Test Mth Test.

This is the last test for series convergence that we're going to be looking at. As with the Ratio Test, this test will also tell whether a series is absolutely convergent or not rather than simple convergence.

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}|a_n|^{\frac{1}{n}}$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

As with the ratio test, if we get $L = 1$ the root test will tell us nothing and we'll need to use another test to determine the convergence of the series. Also note that if $L = 1$ in the Ratio Test then the Root Test will also give $L = 1$.

We will also need the following fact in some of these problems.

Fact

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Let's take a look at a couple of examples.

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$$

Solution

There really isn't much to these problems other than computing the limit and then using the root test. Here is the limit for this problem.

$$L = \lim_{n\to\infty} \left| \frac{n^n}{3^{1+2n}} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+2}} = \frac{\infty}{3^2} = \infty > 1$$

So, by the Root Test this series is divergent

Example 2. Determine if the ff converges or diverges

a. .

$$\sum_{n=1}^{\infty} \left(\frac{3n+1}{4-2n} \right)^{2n}$$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left| \left(\frac{3n+1}{4-2n} \right)^{2n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \left(\frac{3n+1}{4-2n} \right)^{2} \right| = \left(-\frac{3}{2} \right)^2 = \frac{9}{4}$$

we can see that $L = \frac{9}{4} > 1$ and so by the Root Test the series **diverges**

b. $$\sum_{n=0}^{\infty} \frac{n^{1-3n}}{4^{2n}}$$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left| \frac{n^{1-3n}}{4^{2n}} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{n^{\frac{1}{n}-3}}{4^2} \right| = \left| \frac{n^{\frac{1}{n}}\, n^{-3}}{4^2} \right| = \frac{(1)(0)}{16} = 0$$

we can see that $L = 0 < 1$ and so by the Root Test the series converges.

Try

c. $$\sum_{n=4}^{\infty} \frac{(-5)^{1+2n}}{2^{5n-3}}$$

ANS $L = \frac{25}{32} < 1$ and so by the Root Test the series converges.

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$$

$$a_n = \frac{n^n}{3^{1+2n}}$$

$$L = \lim_{n\to\infty} \left| \left(\frac{n^n}{3^1 \cdot 3^{2n}} \right)^{\frac{1}{n}} \right|$$

$$= \left| \frac{n^{n \cdot \frac{1}{n}}}{3^{1 \cdot \frac{1}{n}} \cdot 3^{2n \cdot \frac{1}{n}}} \right|$$

$$\left| \frac{n}{3^{\frac{1}{n}} \cdot 3^2} \right|$$

$$\frac{1}{9} \lim_{n\to\infty} \left| \frac{n}{3^{\frac{1}{n}}} \right| = \frac{1}{9} \cdot \infty = \infty$$

Hence ~~convergent~~. divergent.

$$\sum_{n=0}^{\infty} \frac{n^{1-3n}}{4^{2n}}$$

$$\lim_{n\to\infty} \left(\left| \frac{n^{1-3n}}{4^{2n}} \right| \right)^{\frac{1}{n}}$$

$$\left(\frac{n^{1} \cdot n^{-3n}}{4^{2n}} \right)^{\frac{1}{n}}$$

$$\left| \frac{n^{1 \cdot \frac{1}{n}} \cdot n^{-3n \cdot \frac{1}{n}}}{4^2} \right|$$

$$\left| \frac{n^{\frac{1}{n}} \cdot n^{-3}}{16} \right| = \frac{1}{16} \left| n^{\frac{1}{n}} \cdot n^{-3} \right|$$

$$\frac{1}{16} \left(\infty \right) = 0$$

UNIT 6 INTRODUCTION TO PARTIAL FRACTIONS

Consider the following combination of algebraic fractions:

$$\frac{2}{x-3}-\frac{4}{x-1}=\frac{2(x-1)-4(x-3)}{(x-3)(x-1)}$$

$$=\frac{2x-2-4x+12}{(x-3)(x-1)}$$

$$=\frac{10-2x}{(x-3)(x-1)}$$

$$=\frac{10-2x}{x^2-4x+3}$$

The fractions on the left are called the *partial fractions* of the fraction on the right.

The reverse process of moving from $\frac{10-2x}{x^2-4x+3}$ to $\frac{2}{x-3}-\frac{4}{x-1}$ is called resolving into **partial fractions.**

In order to resolve an algebraic expression into partial fractions:

(i) the denominator of the fraction must factorize (for example, x^2-4x+3 factorizes as $(x-3)(x-1)$).

(ii) the numerator must be at least one degree less than the denominator (in the above example, $10-2x$ is of degree 1 since the highest powered x term is 1 and x^2-4x+3 is of degree 2).

When the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator. We will be looking at basically three types of partial fraction namely; partial fractions with linear factors, partial fractions with repeated factors and partial fractions with quadratic factors.

a. Distinct Linear Factors

In this case, the denominator *Q(x)* can be factored into linear factors, such that, all of them are distinct or different. The decomposition of *Q(x)* is as follows;

$$Q(x)=(x+a_i)(x+a_i)...(x+a_n)$$

Note that no two a_i's are equal, where $i=1,2,...,n$.

Then $\frac{P(x)}{Q(x)}=\frac{A_1}{x+a_1}+\frac{A_2}{x+a_2}+...+\frac{A_n}{x+a_n}$, where $A_1, A_2,..., A_n$ are constants.

Examples:

1. Express $\frac{2x+5}{x^2-x-2}$ in partial fractions.

   Solution:

   Firstly, the denominator is factorized to give:

$$\frac{2x+5}{x^2-x-2}=\frac{2x+5}{(x-2)(x+1)}$$

Resolve the fraction to partial fraction

$$\frac{2x+5}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$$

$$\Rightarrow 2x+5=A(x+1)+B(x-2)$$

If $x=-1, \Rightarrow 2(-1)+5=B(-1-2)$

$$3=-3B$$

$\therefore B=-1$

If $x=2, \Rightarrow 2(2)+5=A(2+1)$

$$9=3A$$

$\therefore A=3$

Hence, $\dfrac{2x+5}{x^2-x-2}=\dfrac{3}{x-2}-\dfrac{1}{x+1}$

NB: Other methods can be used to find A and B such as the "cover up method" and the "comparison of coefficients".

2. If $y=\dfrac{2x-3}{(x^2-1)(x+2)}$, express y in partial fractions.

Solution:

$$\frac{2x-3}{(x^2-1)(x+2)} = \frac{2x-3}{(x-1)(x+1)(x+2)}$$

$$\frac{2x-3}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$$

$$\Rightarrow 2x-3 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$

$$\text{If } x=-1, \Rightarrow 2(-1)-3 = B(-1-1)(-1+2)$$

$$-5 = -2B$$

$$\therefore B = \frac{5}{2}$$

$$\text{If } x=1, \Rightarrow 2(1)-3 = A(1+1)(1+2)$$

$$-1 = 6A$$

$$\therefore A = -\frac{1}{6}$$

$$\text{If } x=-2, \Rightarrow 2(-2)-3 = C(-2-1)(-2+1)$$

$$-7 = 3C$$

$$\therefore C = \frac{-7}{3}$$

$$\text{Hence, } y = \frac{-1}{6(x-1)} + \frac{5}{2(x+1)} - \frac{7}{3(x+2)}$$

b. Repeated Linear Factors

Here $Q(x)$ which is the denominator can be factored into repeated linear factors, that is,

$Q(x) = (x+a_n)^{r1}(x+a_n)^{r2}...(x+a_n)^{rn}$.

The following examples will help elaborate how this principle works.

i. $\frac{2x+3}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$

ii. $\frac{5x^2+20x+6}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

Examples:

1. Express $\frac{2x+3}{(x-2)^2}$ in partial fractions.

Solution:

$$\frac{2x+3}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$=\frac{A(x-2)+B}{(x-2)^2}$$

$\Rightarrow 2x+3=A(x-2)+B$

If $x=2, \Rightarrow 2(2)+3=B$

$\therefore B=7$

$\Rightarrow 2x+3=A(x-2)+7$

Since an identity is true for all values of the unknown, the coefficients of similar terms may be equated.

Hence, equating the coefficients of x gives: $2=A$

$\therefore A=2$

Hence, $\dfrac{2x+3}{(x-2)^2}=\dfrac{2}{x-2}+\dfrac{7}{(x-2)^2}$

2. Resolve $\dfrac{5x^2-2x-19}{(x+3)(x-1)^2}$ into partial fractions.

Solution:

$2A-B=2$ — (3)

$A+B=2$ — (2)

$\frac{2x+7}{(x-2)^3}=\frac{2}{(x-2)^2}+\frac{7}{(x+2)^3}$

$\frac{5x^2-2x-19}{(x+3)(x-1)^2}=\frac{A}{(x+3)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$

$5x^2-2x-19=\frac{A(x-1)^2+B(x-1)(x+3)+C(x+3)}{(x+3)(x-1)^2}$

Put $x=-3$

$5(-3)^2-2(-3)-19=A(-3-1)^2+B(-3-1)(-3+3)+C(-3+3)$

$45+6-19=16A+0$

$\frac{32}{16}=\frac{16A}{16}$

$A=2$

Put $x=1$

$5-2-19=4C$

$\frac{-16}{4}=\frac{4}{4}C$

$C=-4$

Put $x=0$

$-19=A-3B+3C$

$-19=2-3B-12$

$3B=-12+19+2$

$\frac{3B}{3}=\frac{9}{3}$

$B=3$

$\frac{5x^2-2x-19}{(x+3)(x-1)^2}=\frac{2}{(x+3)}+\frac{3}{(x-1)}$

$$\frac{5x^2-2x-19}{(x+3)(x-1)^2}=\frac{A}{x+3}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$=\frac{A(x-1)^2+B(x+3)(x-1)+C(x+3)}{(x+3)(x-1)^2}$$

Equating the numerators gives:

$5x^2-2x-19=A(x-1)^2+B(x+3)(x-1)+C(x+3)$

If $x=-3,\Rightarrow 5(-3)^2-2(-3)-19=A(-3-1)2$

$32=16A$

$\therefore A=2$

If $x=1,\Rightarrow 5(1)^2-2(1)-19=C(1+3)$

$-16=4C$

$\therefore C=-4$

Now, comparing coefficients, $5x^2=Ax^2+Bx^2$

$\Rightarrow 5=A+B$

$5=2+B$

$\therefore B=3$

Thus, $\frac{5x^2-2x-19}{(x+3)(x-1)^2}=\frac{2}{x+3}+\frac{3}{x-1}-\frac{4}{(x-1)^2}$

c. Quadratic Factors

The denominator is a quadratic factor which does not factorize without introducing imaginary surd terms. Hence Q(x) which is the denominator can only be factored into quadratic expression. That is, $Q(x)=(x^2+b_1x+c_1)+(x^2+b_2x+c_2)+...(x^2+b_nx+c_n)$.

The following examples will help elaborate how this principle works.

i. $\frac{2x+1}{x^2+2x+5}=\frac{Ax+B}{x^2+2x+5}$

ii. $\frac{3x^2+3x-7}{(x^2+5)(x^2+3x+20)}=\frac{Ax+B}{x^2+5}+\frac{Cx+D}{x^2+3x+20}$

Examples:

1. Express $\frac{5x^2+7x+8}{(x+1)(x^2+2x+3)}$ in partial fractions.

   Solution:

$$\frac{5x^2+7x+8}{(x+1)(x^2+2x+3)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+3}$$

$$=\frac{A(x^2+2x+3)+(Bx+C)(x+1)}{(x+1)(x^2+2x+3)}$$

$$5x^2+7x+8=A(x^2+2x+3)+(Bx+C)(x+1)$$

If $x=-1,\Rightarrow 5(-1)^2+7(-1)+8=A((-1)^2+2(-1)+3)$

$$6=2A$$

$\therefore A=3$

If $x=0,\Rightarrow 8=3A+C$

$$8=3(3)+C$$

$\therefore C=-1$

If $x=1,\Rightarrow 5(1)^2+7(1)+8=3(1^2+2(1)+3)+(B(1)-1)(1+1)$

$$20=18+2B-2$$

$\therefore B=2$

Thus, $\frac{5x^2+7x+8}{(x+1)(x^2+2x+3)}=\frac{3}{x+1}+\frac{2x-1}{x^2+2x+3}$

2. Express $\frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}$ in partial fractions.

Solution:

$$\frac{3+6x+4x^2-2x^2}{x^2(x^2+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3}$$

$$= \frac{Ax(x^2+3)+B(x^2+3)+(Cx+D)x^2}{x^2(x^2+3)}$$

Equating the numerators gives:

$$3+6x+4x^2-2x^2 = Ax(x^2+3)+B(x^2+3)+(Cx+D)x^2$$

$$= Ax^2+3Ax+Bx^2+3B+Cx^3+Dx^2$$

If $x=0$, then, $3=3B$

$$\Rightarrow B=1$$

Now, equating the coefiients of x^2 terms gives:

$4=B+D \Rightarrow 4=1+D$

$\therefore D=3$

Equating the coefiients of x terms gives:

$6=3A \Rightarrow A=2$

Equating the coefiients of x^3 terms gives:

$-2=A+C \Rightarrow -2=2+C$

$\therefore C=-4$

Hence, $\dfrac{3+6x+4x^2-2x^2}{x^2(x^2+3)} = \dfrac{2}{x} + \dfrac{1}{x^2} + \dfrac{-4x+3}{x^2+3}$

$$= \frac{2}{x} + \frac{1}{x^2} + \frac{3-4x}{x^2+3}$$

NB: In the case of repeated quadratic factors, combine the methods used in repeated linear factors and quadratic factors to resolve into partial fractions.

For example; $\dfrac{2x+3}{(x^2+4)^2} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2}$

Recall in order to resolve an algebraic expression into partial fractions, the numerator must be at least one degree less than the denominator. However when the degree of the numerator is equal to or higher than the degree of the denominator, the numerator must be divided by the denominator until the remainder is of less degree than the denominator.

For example;

Express $\dfrac{x^2+3x-10}{x^2-2x-3}$ in partial fractions.

Solution:

$$\frac{x^2+3x-10}{x^2-2x-3} = \frac{x^2-2x-3+5x-7}{x^2-2x-3}$$

$$= 1 + \frac{5x-7}{x^2-2x-3}$$

$$= 1 + \frac{5x-7}{(x+1)(x-3)}$$

Self Assessment.

Express the following into partial fractions.

1. $\dfrac{2x^2-4x+3}{(x-2)(x+1)}$ $\left(\text{Did you get } \dfrac{2x^2-4x+3}{(x-2)(x+1)} = 2+\dfrac{1}{x-2}-\dfrac{3}{x+1}?\right)$.

2. $\dfrac{5}{(x+1)(x-2)}$

3. $\dfrac{2x^2+3x+3}{(x+3)(x+2)x}$

4. $\dfrac{2x}{(x^3-8)}$

5. $\dfrac{x^3+1}{(x-1)^2}$

UNIT 7
MATHEMATICAL INDUCTION

Mathematical induction is the method of proof frequently used to prove general formulae, such as a formula for the sum of a sequence of n numbers.
This method consists of three major steps.

1. Verify that the proposed formula is true for an initial (small) value of n. That is to show that it is true for $n = 1$
2. While assuming that the proposed formula is true for a specific value of n, prove that the formula is also true for the next value of n. That is, show that if $n = k$ is true then $n = k+1$ is also true.
3. Conclude that (because of mathematical induction) the formula in fact does hold for all values of n.

Examples:

1. Prove by mathematical induction that $1+2+3+4+...+n=\frac{n(n+1)}{2}$, for all natural numbers.

Solution:

Let S(n) be the statement

The statement is true for S(1) (i.e. n =1), since $1=\frac{1(1+1)}{2}=1$

Assume S(k) is also true (i.e. $n = k$)

Thus, $1+2+3+4+...+k=\frac{k(k+1)}{2}$

If S(k) is true, the S(k+1) is also true (i.e. If the statement is true for $n = k$, then it is true for $n = k+1$)

Thus, $1+2+3+4+...+k+(k+1)=\frac{k(k+1)}{2}$

$$\text{Thus, } 1+2+3+4+...+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$
$$=\frac{k(k+1)+2(k+1)}{2}$$
$$=\frac{(k+1)(k+2)}{2}$$
$$=\frac{(k+1)[(k+1)+1]}{2}$$

Hence for all natural numbers $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$.

2. Prove by mathematical induction that $1^2+2^2+3^2+4^2+...+n^2=\frac{1}{6}n(n+1)(2n+1)$, for all natural numbers.

Solution:

Let p(n) be the statement

The statement is true for p(1), since $1^2 = \frac{1}{6}(1)(2)(3) \Rightarrow 1=1$

Assume p(k) is also true (i.e. $n = k$)

Thus, $1^2 + 2^2 + 3^2 + 4^2 + ... + k^2 = \frac{1}{6}k(k+1)(2k+1)$

If p(k) is true, the p(k+1) is also true

Hence, $1^2 + 2^2 + 3^2 + 4^2 + ... + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1)$

Implying that $1^2 + 2^2 + 3^2 + 4^2 + ... + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

$$= \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]$$

Hence for all natural numbers $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

3. Prove by mathematical induction that $\sum_{j=1}^{n} j^3 = \frac{1}{4}n^2(n+1)^2$.

Solution:

Let P(n) be $\sum_{j=1}^{n} j^3 = \frac{1}{4}n^2(n+1)^2$.

For n=1,

$P(1) = 1^3 = \frac{1}{4}1^2(1+1)^2 = \frac{4}{4} \quad \Rightarrow 1 = 1$ is true.

Assuming the formula is true for $n = k$. That is,

$P(k) \Rightarrow \sum_{r=1}^{n} k^3 = \frac{1}{4}k^2(k+1)^2$

If P(k) is true, the P(k+1) is also true. That is,

$$P(k+1) \Rightarrow \sum_{r=1}^{n} k^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2$$

$$\Rightarrow \sum_{r=1}^{n} k^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2[k^2 + 4(k+1)]$$

$$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \frac{1}{4}(k+1)^2[(k+1)+1]^2$$

Hence for all natural numbers $\sum_{j=1}^{n} j^3 = \frac{1}{4}n^2(n+1)^2$.

NB: Example 3 can be stated differently as prove by mathematical induction that $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2$.

4. Prove by mathematical induction that $1+3+5+...+(2n-1) = n^2$, for all natural numbers.

Solution:
Let p(n) be the statement
The statement is true for p(1) (i.e. $n=1$),, since $2(1)-1=1^2 \Rightarrow 1=1$
Assume p(k) is also true (i.e. $n=k$)
Thus, $1+3+5+...+(2k-1)=k^2$
If p(k) is true, the p(k+1) is also true
Hence, $1+3+5+...+(2k-1)+[2(k+1)-1]=k^2$
Implying that, $1+3+5+...+(2k-1)+[2(k+1)-1] = k^2 + [2(k+1)-1]$

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$

Hence the formula is true for all positive integral values of n by induction.

NB: The above formula can be stated as "prove by mathematical induction that the sum of the first n odd numbers is equal to the nth square number".

Self Assessment

1. Prove by mathematical induction that $\frac{1}{1.2}+\frac{1}{2.3}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for all integers $n\geq 1$
2. Use mathematical induction to prove that $S_n=2+4+6+8+\cdots+2n=n(n+1)$ for every positive integer n.

1 A
2 B
3 C
4 A
5 A
6 D
7 C
8 B
9 A
10 D

11 C
12 D
13 B
14 C
15 C
16 C
17 B
18 A
19 C
20 D

21 C
22 A
23 C
24 B
25 D
26 C
27 B
28 C
29 C
30 D